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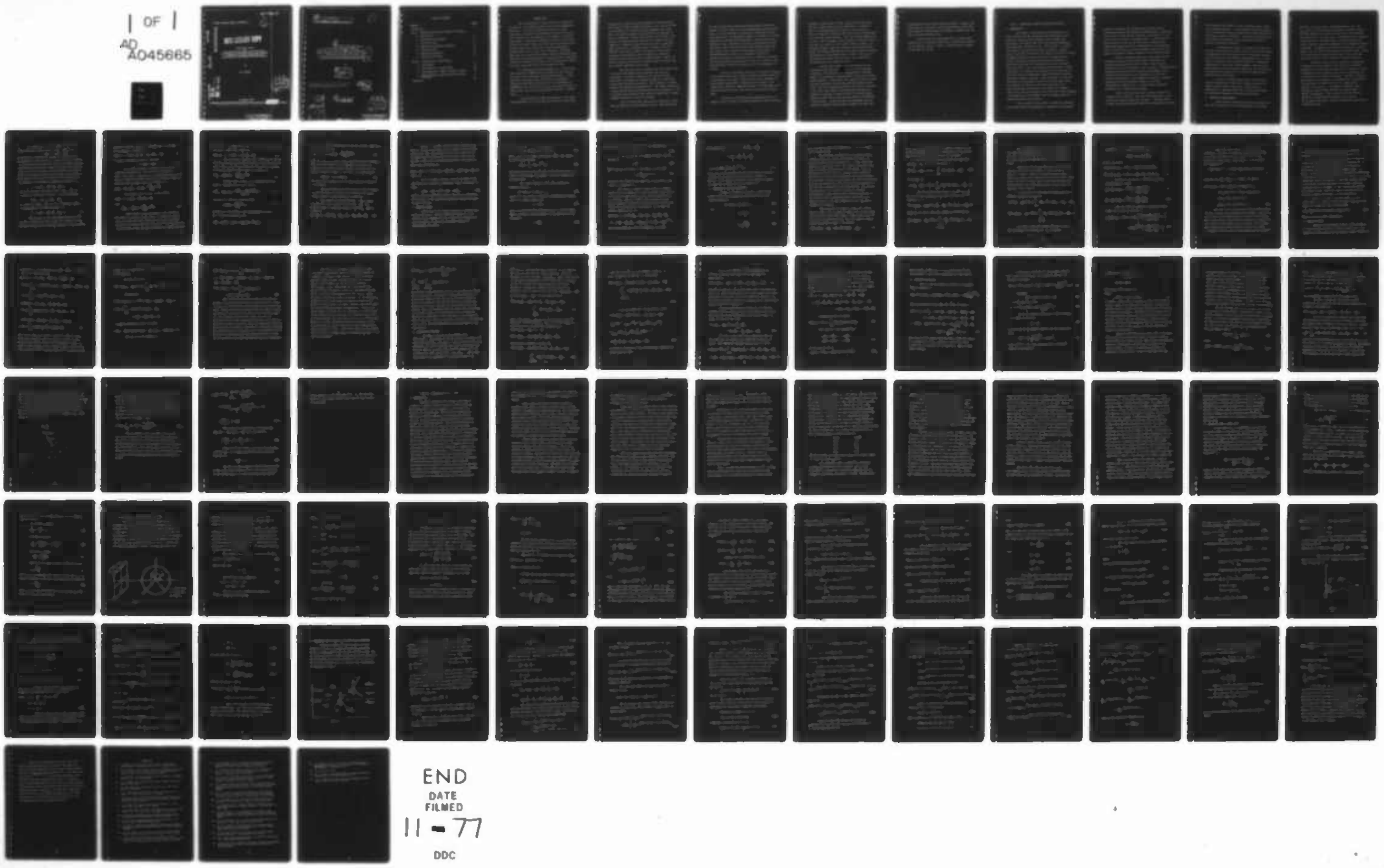

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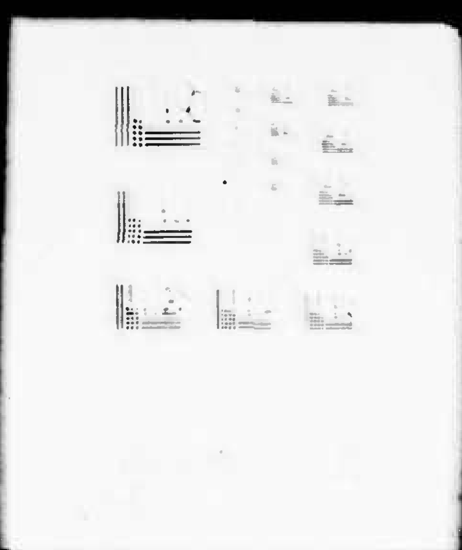

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Preliminary Report
on
The Mathematical Theory of the Multiple Scattering of an Acoustic Pulse from a Random Collection of Volume Scatterers with Application to Scattering from Fish Schools

by

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31 August 1969

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INTRODUCTION

The classification of sonar contacts as submarine or non-submarine is a primary concern to Naval forces assigned an ASW mission. Incorrect classification results in undue expenditure of resources to investigate or conduct an attack on a non-submarine, or in permitting a true submarine to proceed unchallenged. Though the classification problem at short sonar ranges has not been solved, the effects and sources giving rise to short range false contacts have received considerable attention and, in most cases, can be described. Sufficient experience in the use of Bottom Bounce (BB) and Convergence Zone (CZ) modes of propagation has not been accumulated to adequately evaluate the character and extent of the classification problem with long range sonars using these propagation paths.

A study was initiated at TRACOR, Inc., in March 1969 under Contract No. N00024-69-C-1255 to evaluate the character and extent of the long range classification problem. It was decided to investigate first the BB mode of propagation, reserving the CZ mode for future work. An investigation by TRACOR[1] has shown that schools of fish are probable sources of false contacts for sonars operating with a CW pulse in the BB mode.

For the purpose of that investigation a false target was considered to be any non-submarine producing an echo whose

characteristics caused it to be classified as a submarine echo on the basis of echo strength and duration. For a school of fish to be mistaken for a submarine several conditions must exist. First, the target strength of the school must be comparable to that of a submarine. Second, the extension of the school in the direction of sound propagation should not be greater than the length of a submarine so that the echo will not be longer than that expected from a legitimate target. Finally, since the azimuthal beam width at ranges of interest in bottom bounce operations is much greater than the length of a submarine, detection on more than one beam is highly unlikely. Hence, the school of fish should subtend an azimuthal angle no greater than the azimuthal beam width of the sonar.

Ray theory was used to determine the volume which could be occupied by the fish school consistent with the above constraints. The target strength for the school was then calculated by multiplying the target strength of a single fish by the number of fish in the volume. This procedure, of course, neglects all multiple scattering effects. The target strength of a single fish was taken from measurements made by Cushing, et. al.[2] It was found that schools of fish could exist which have a target strength comparable to that of a submarine.

Now it is known that some schools of fish are so dense that the fish are in actual physical contact. One would expect

for such a school that multiple scattering effects could not be ignored. If this is the case, then classification characteristics such as (1) echo duration, (2) spectrum, (3) sharpness of onset and trailing edge, (4) highlight structure, (5) doppler, (6) target strength and (7) ping-to-ping consistency and persistence would differ from those calculated on the basis of neglecting multiple scattering. Thus, since this initial investigation showed that fish schools posed a problem to long range classification, it was decided that the only way to adequately evaluate the character and extent of this problem was to formulate a realistic model of the multiple scattering of a pulse from a school of fish. This preliminary report takes the first step in developing such a model.

In Part I, the general theory of the multiple scattering of an acoustic pulse from a random collection of point scatterers is developed in some detail. Only the first order statistics of the scattered signal will be considered in this report. The second order statistics, which are related to the scattering cross section of the fish school, will have to be reserved for future work.

In Part II, the general theory developed in Part I is applied to the case where the random collection of point scatterers

represents a dense school of fish. The scattering properties used for a single fish are those reported by D. E. Weston[3]. Thus, it is assumed that the scattering is due entirely to the air bladder of the fish and that the fish can be represented as an isotropic point scatterer. This will be approximately the case for a fish of length L if the frequency of the sound wave is less than c_o/L, where c_o is the speed of sound in water. For frequencies greater than c_o/L scattering from the fish tissue becomes increasingly important and the simple theory of isotropic scattering from an air bladder does not apply. The theory of scattering from bladder fish takes into account dispersion and absorption in the scattering volume.

Two cases are considered for the geometry of the scattering region simply because they are the most amenable to solution: (1) the fish are contained within a spherical region; (2) the fish are contained within a layer with infinite plane boundaries. In the first case, spherical schools of fish have been observed in the open sea, but the diameter of the schools and the distribution of fish in them were not reported. In the second case, while a layer with plane boundaries is amenable to solution, it does not meet all the requirements imposed on fish schools which were discussed at the beginning of this section. It does not meet the requirement that the school be contained entirely within a single beam since the

plane boundaries of the layer extend to infinity. However, this would serve as a good model for the Deep Scattering Layer (DSL) since there is a growing amount of evidence to indicate that the DSL is composed of bladder fish.

At this time it is anticipated that the first order statistics for the scattered field will be numerically evaluated for the spherical school. However, no further work is planned on the model for the DSL.

Part I. MATHEMATICAL THEORY OF MULTIPLE SCATTERING

A. INTRODUCTION

The problem of scattering of an incident wave by a single obstacle has been considered rather thoroughly in the literature, beginning with Rayleigh's work in fluids, and continuing to the present day with various quantum-mechanical, electromagnetic, and elastic cases. The literature on multiple scattering, on the other hand is not so extensive. Regular arrays of scatterers were treated by Huygen's principle or various perturbation schemes, mainly with an eye to obtaining their "strong" filtering properties arising due to periodicity. Out of this work came X-ray diffraction theory and the band theory of solids. Until 1945 multiple scattering from a random distribution of scatterers was restricted to the scattering of particles based on the Boltzmann integro differential equation describing transport processes. This formulation is merely the expression of conservation of particles in phase space; hence the treatment is classical, with no account taken of the quantum-mechanical wave nature of the particles or photons. Such a theory would be expected to be valid only if the wavelength of the particles is much smaller than the average distance of separation between the scatterers.

There are also a large number of problems of multiple scattering in which the wavelength is comparable to the average

scatterer separation; some examples of this latter type of problem are acoustic wave propagation in bubbly water, scattering from a school of fish and the scattering of electrons or x-rays by the nuclei of liquids or amorphous solids. Any treatment of these problems must include the reflection, refraction, and interference phenomena that are characteristic of wave problems; hence it must be based on the wave equation, rather than on the simple conservation statement leading to the Boltzmann equation.

The first systematic treatment of the problem of the multiple scattering of waves from a random distribution of scatterers was published by Foldy[4] in 1945. Foldy's unique contribution was the introduction of the concept of "configurational" averaging of relevant physical quantities by defining a joint probability distribution for the occurrence of a particular scatterer configuration. By averaging the equations of multiple scattering over the statistical ensemble of scatterer configurations, Foldy was able to derive integral equations governing these configurational averages.

Foldy's work dealt only with a random collection of isotropic point scatterers. This was later generalized by Lax[5] to treat the multiple scattering of quantum mechanical waves by point scatterers having quite general scattering characteristics,

and by Waterman and Truell[6] to treat scattering regions having non-vanishing dimensions. A unified picture of the propagation of the coherent and incoherent radiation associated with multiple scattering is given by McCloskey[7]. Surveys of the entire field of the multiple scattering of waves have been published by Twersky[8], and Burke and Twersky[9].

To the best of the author's knowledge, the theory of multiple scattering from a random collection of scatterers which has appeared in the literature to date has only been concerned with the scattering of an incident monochromatic wave. In this report an attempt is made to extend the theory to include the scattering of a pulse (time limited signal). Only the first order statistics of the scattered signal will be considered here. The second order statistics of the scattered signal will be considered in a later report.

In Part I a detailed description of the theory of multiple scattering of a pulse from a random collection of volume scatterers will be presented. Results found in References 4-7, mentioned above, will be freely used without any additional references to these papers being made.

B. STATISTICAL PRELIMINARIES

Assume we have a collection of N point scatterers and that the i^{th} scatterer can be characterized by its position

vector $\vec{r}_i = (x_i, y_i, z_i)$ and a scattering parameter a_i. Here x_i, y_i and z_i are the Cartesian coordinates of the i^{th} scatterer. The only application that will be considered is the scattering from bladder fish. For this case it is assumed that the scattering is due entirely to the swim bladder of the fish and that the actual elongated bladder can be replaced by an equivalent spherical air bladder. Consequently, the scattering parameter a_i will be taken to be the radius of the equivalent spherical bladder. There is no difficulty in generalizing the theory if more than one scattering parameter must be associated with each scatterer.

We shall say that we have a particular configuration of scatterers if the position vectors $\vec{r}_i$ and the scattering parameters a_i are specified for each of the N scatterers. This configuration may be regarded as one state in an ensemble and an average over all states may be taken. In order to simplify the notation in what follows, a four dimensional vector $\vec{q}_i$ will be introduced and defined to have as its components $\vec{q}_i = (\vec{r}_i, a_i) = (x_i, y_i, z_i, a_i)$. Also, $d\vec{q}_i$ will represent a four dimensional volume element and will be defined as $d\vec{q}_i = d\vec{r}_i da_i = dx_i dy_i dz_i da_i$. Thus, we can now say that we have a particular configuration of scatterers if the vectors $\vec{q}_i$ $(1 \leq i \leq N)$ are specified. This four dimensional space will be called phase space.

To further specify this ensemble, we introduce a joint probability density function $p(\vec{q}_1,\ldots,\vec{q}_N)$ so that

$$p(\vec{q}_1,\ldots,\vec{q}_N)d\vec{q}_1\ldots d\vec{q}_N$$

represents the probability of finding the scatterers in a configuration in which the first scatterer lies in an element of volume $d\vec{r}_1$ about the point $\vec{r}_1$ and has a scattering parameter lying between a_1 and a_1+da_1, while at the same time the second scatterer lies in an element of volume $d\vec{r}_2$ about the point $\vec{r}_2$ and has a scattering parameter lying between a_2 and a_2+da_2, etc. The integral of the joint probability distribution over all configurations of scatterers is normalized to unity; i.e.,

$$\int\ldots\int p(\vec{q}_1,\ldots,\vec{q}_N)\, d\vec{q}_1\ldots d\vec{q}_N = 1\ . \qquad (1)$$

We can also introduce the number density $n(\vec{q}_1,\ldots,\vec{q}_N)$ of scatterers by multiplying through Eq. 1 by N and letting

$$n(\vec{q}_1,\ldots,\vec{q}_N) = N\, p(\vec{q}_1,\ldots,\vec{q}_N)\ . \qquad (2)$$

The integral of $n(\vec{q}_1,\ldots,\vec{q}_N)$ over all of phase space gives the total number of scatterers present; i.e.,

$$\int\ldots\int n(\vec{q}_1,\ldots,\vec{q}_N)\, d\vec{q}_1\ldots d\vec{q}_N = N\ . \qquad (3)$$

The probability of a configuration having the i^{th} scatterer occupying the volume element $d\vec{q}_i$ regardless of the location and value of the scattering parameter of all the other

scatterers, may be obtained by integrating over all but the i^{th} coordinate as follows:

$$p(\vec{q}_i) = \int \cdots \int p(\vec{q}_1 \cdots \vec{q}_N)\, d\vec{q}_1 \cdots d\vec{q}_{i-1} d\vec{q}_{i+1} \cdots d\vec{q}_N \ . \qquad (4)$$

In terms of the number density we can write

$$N\, p(\vec{q}_i) = n(\vec{q}_i) \ . \qquad (5)$$

It is also convenient to introduce the notion of conditional probabilities. The conditional probability, $p(\vec{q}_1,\ldots,\vec{q}_{i-1}, \vec{q}_{i+1},\ldots,\vec{q}_N | \vec{q}_i)$, for a configuration having the i^{th} scatterer fixed at the known location $\vec{r}_i$ and having a scattering parameter with the known value a_i is defined to be

$$p(\vec{q}_1,\ldots,\vec{q}_{i-1},\vec{q}_{i+1},\ldots,\vec{q}_N | \vec{q}_i) = \frac{p(\vec{q}_1,\ldots \vec{q}_N)}{p(\vec{q}_i)} \ . \qquad (6)$$

For example, if i = 1, Eqs. 4 and 6 become

$$p(\vec{q}_1) = \int \cdots \int p(\vec{q}_1,\ldots,\vec{q}_N)\, d\vec{q}_2 \cdots d\vec{q}_N \ , \qquad (7)$$

and

$$p(\vec{q}_2,\ldots, \vec{q}_N | \vec{q}_1) = \frac{p(\vec{q}_1,\ldots,\vec{q}_N)}{p(\vec{q}_1)} \ . \qquad (8)$$

These notions can be generalized to the case where more than one scatterer is held fixed. To facilitate writing let us take the case where scatterers 1 and 2 are held fixed instead of the more general case where the i^{th} and j^{th} scatterers are held fixed. First, the joint probability of finding the first

scatterer in the volume element $d\vec{q}_1$ and the second scatterer in the volume element $d\vec{q}_2$ is given by

$$p(\vec{q}_1,\vec{q}_2) = \int\cdots\int p(\vec{q}_1,\ldots,\vec{q}_N)\, d\vec{q}_3\ldots d\vec{q}_N \; . \tag{9}$$

The conditional probability for a configuration having the first scatterer fixed at $\vec{r}_1$ and having a scattering parameter with a value a_1 and the second scatterer fixed at $\vec{r}_2$ and having a scattering parameter with a value a_2 is defined as

$$p(\vec{q}_3,\ldots,\vec{q}_N|\vec{q}_1,\vec{q}_2) = \frac{p(\vec{q}_1,\ldots,\vec{q}_N)}{p(\vec{q}_1,\vec{q}_2)} \; . \tag{10}$$

Another useful relation can be obtained from these results. Combining Eqs. 8 and 10 gives

$$p(\vec{q}_3,\ldots,\vec{q}_N|\vec{q}_1,\vec{q}_2) = \frac{p(\vec{q}_2,\ldots,\vec{q}_N|\vec{q}_1)p(\vec{q}_1)}{p(\vec{q}_1,\vec{q}_2)} \; . \tag{11}$$

Moreover, Eq. 8 for the case N=2 yields

$$p(\vec{q}_2|\vec{q}_1) = \frac{p(\vec{q}_1,\vec{q}_2)}{p(\vec{q}_1)} \; . \tag{12}$$

Solving Eq. 12 for $p(\vec{q}_1,\vec{q}_2)$ and substituting this result in Eq. 11 gives the desired result

$$p(\vec{q}_2,\ldots,\vec{q}_N|\vec{q}_1) = p(\vec{q}_3,\ldots,\vec{q}_N|\vec{q}_1,\vec{q}_2)\; p(\vec{q}_2|\vec{q}_1) \; . \tag{13}$$

As a final illustration of the use of number densities, we can write Eq 12 as

$$p(\vec{q}_1,\vec{q}_2) = p(\vec{q}_1)p(\vec{q}_2|\vec{q}_1) = \frac{n(\vec{q}_1)n(\vec{q}_2|\vec{q}_1)}{N(N-1)} \ . \qquad (14)$$

Here the conditional number density $n(\vec{q}_2|\vec{q}_1)$ has been introduced. The conditional number density $n(\vec{q}_2|\vec{q}_1)$ is related to the conditional probability density $p(\vec{q}_2|\vec{q}_1)$ in the same way that $n(\vec{q}_1)$ is related to $p(\vec{q}_1)$ (cf. Eq. 5); i.e., by the relation

$$n(\vec{q}_2|\vec{q}_1) = (N-1)\ p(\vec{q}_2|\vec{q}_1) \ . \qquad (15)$$

The factor (N-1) appears in Eq. 15 simply because one particle is held fixed.

The generalization of these results to the case where an arbitrary number of scatterers are held fixed with known values of their scattering parameter is immediate.

Now consider a function $f(\vec{r},t|\vec{q}_1,\ldots,\vec{q}_N)$ of space coordinates $\vec{r}$, time t and the N vectors $\vec{q}_1,\ldots,\vec{q}_N$. The configurational (or ensemble) average of $f(\vec{r},t|\vec{q}_1,\ldots,\vec{q}_N)$ is defined by

$$\left\langle f(\vec{r},t)\right\rangle = \int\ldots\int f(\vec{r},t|\vec{q}_1,\ldots,\vec{q}_N)\ p(\vec{q}_1,\ldots,\vec{q}_N)\ d\vec{q}_1\ldots d\vec{q}_N \ . \qquad (16)$$

The partial average with one or more scatterers held fixed is obtained by averaging over the appropriate conditional probability: Thus, the average of $f(\vec{r},t \mid \vec{q}_1,\ldots,\vec{q}_N)$ over all configurations for which the first scatterer is held fixed at $\vec{r}_1$ with a value a_1 for its scattering parameter is denoted by $\langle f(\vec{r},t \mid \vec{q}_1)\rangle$ and defined by

$$\langle f(\vec{r},t \mid \vec{q}_1)\rangle = \int\ldots\int f(\vec{r},t \mid \vec{q}_1,\ldots,\vec{q}_N)p(\vec{q}_2,\ldots,\vec{q}_N \mid \vec{q}_1)d\vec{q}_2\ldots d\vec{q}_N, \quad (17)$$

and so on.

If it is assumed that the locations and radii of the scatterers are statistically independent, then by definition we have

$$p(\vec{q}_1,\ldots,\vec{q}_N) = p(\vec{q}_1)\, p(\vec{q}_2)\ldots\, p(\vec{q}_N) \, , \quad (18)$$

that is, the joint probability density function is expressible as the product of the individual probability density functions for each scatterer.

C. ACOUSTIC PULSES

Let $\Psi(\vec{r},t)$ be the acoustic pressure at the space point $\vec{r}$ and at the time t. We will assume that $\Psi(\vec{r},t)$ is a real function of its arguments $\vec{r}$ and t. Since we are interested in pulse wave forms, we shall use the Fourier integral [10] representation for $\Psi(\vec{r},t)$.

That is, we let

$$\Psi(\vec{r},t) = \frac{1}{\pi} \mathrm{Re} \int_{0}^{\infty} \psi(\vec{r},\omega)\, e^{-i\omega t}\, d\omega \,, \tag{19}$$

where $\psi(\vec{r},\omega)$ is the Fourier transform of $\Psi(\vec{r},t)$ and is given by the inversion formula

$$\psi(\vec{r},\omega) = \int_{-\infty}^{\infty} \Psi(\vec{r},t)\, e^{i\omega t}\, dt \,. \tag{20}$$

The symbol Re denotes the real part of the integral which is a complex number and the real quantity $\omega/2\pi$ is the frequency of the sound wave.

The Fourier transform $\psi(\vec{r},\omega)$ must satisfy the Helmholtz equation

$$\nabla^2 \psi(\vec{r},\omega) + k^2 \psi(\vec{r},\omega) = 0 \,, \tag{21}$$

where ∇^2 is the Laplacian operator.[11] If absorption is present, then k is a complex number, say

$$k = \kappa + i\alpha \,, \tag{22}$$

where α is the absorption coefficient and κ is the wave number which is related to the wavelength λ of the sound wave by the expression

$$\kappa = \frac{2\pi}{\lambda} \,. \tag{23}$$

The phase velocity c_p of the sound wave is then given by

$$c_p = \frac{\omega}{\kappa} . \tag{24}$$

In analogy with Eq. 24 a complex velocity c can be defined by

$$c = \frac{\omega}{k} . \tag{25}$$

Upon using Eqs. 22 and 24, we can express c in the form

$$c = \frac{c_p}{1 + i\alpha c_p/\omega} . \tag{26}$$

In general the medium will be dispersive, so that both c_p and α will be a function of ω.

Now consider a medium in which there are embedded N point scatterers which are randomly distributed with respect to position and with respect to the scattering parameter a_i. Let $\Psi(\vec{r},t|\vec{q}_1,\ldots,\vec{q}_N)$ be a scalar field (acoustic pressure field) produced by the multiple scattering of an incident wave $\Psi^i(\vec{r},t)$ by a configuration of scatterers. Let us take the ensemble average of Eq. 19.

$$\langle\Psi(\vec{r},t)\rangle = \int\ldots\int \Psi(\vec{r},t|\vec{q}_1,\ldots,\vec{q}_N)\, p(\vec{q}_1,\ldots,\vec{q}_N)\, d\vec{q}_1\ldots d\vec{q}_N$$

$$= \int\ldots\int d\vec{q}_1\ldots d\vec{q}_N\, p(\vec{q}_1,\ldots,\vec{q}_N)\frac{1}{\pi}\mathrm{Re}\int_0^\infty \psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_N)e^{-i\omega t}\, d\omega .$$

Upon interchanging the order of integration, we obtain

$$\left\langle \Psi(\vec{r},t) \right\rangle = \frac{1}{\pi} \mathrm{Re} \int_0^\infty e^{-i\omega t} \, d\omega \int \ldots \int \psi(\vec{r},\omega | \vec{q}_1, \ldots, \vec{q}_N)$$

$$\times \, p(\vec{q}_1, \ldots, \vec{q}_N) \, d\vec{q}_1 \ldots d\vec{q}_N$$

or

$$\left\langle \Psi(\vec{r},t) \right\rangle = \frac{1}{\pi} \mathrm{Re} \int_0^\infty \left\langle \psi(\vec{r},\omega) \right\rangle e^{-i\omega t} \, d\omega \, . \tag{27}$$

Consequently in developing the theory of multiple scattering, we need only be concerned with the Fourier components $\langle\psi(\vec{r},\omega)\rangle$. A more rigorous discussion of the Fourier transforms of stochastic processes is given by Papoulis[12].

D. EQUATIONS OF MULTIPLE SCATTERING

Consider a homogeneous, isotropic, non-absorbing medium capable of sustaining wave motion according to the Helmholtz equation

$$\nabla^2 \psi(\vec{r},\omega) + \kappa_o^2 \psi(\vec{r},\omega) = o \tag{28}$$

where

$$\kappa_o = \frac{\omega}{c_o} \, , \tag{29}$$

$$c_o = \sqrt{\frac{B_{ad}}{\rho_o}} \, , \tag{30}$$

B_{ad} being the adiabatic bulk modulus and ρ_o the mass density of the liquid medium.

Suppose there are embedded in this medium N point scatterers which are randomly distributed with respect to position $\vec{r}_m$ and scattering parameter a_m $(1 \leq m \leq N)$. Under the influence of an incident wave $\psi^i(\vec{r},\omega)$ and the scattering from other scatterers, a scattered wave $\psi^s(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N)$ is generated by the m^{th} scatterer. Here the first argument $\vec{r}$ specifies the field point of evaluation, the second argument ω is the angular frequency of the incident and scattered radiation, $\vec{q}_m$ gives the location and value of the scattering parameter of the scatterer originating the radiation and $\vec{q}_1,\ldots,\vec{q}_N$ indicate the dependence of the scattered wave on the specific configuration chosen. Employing the radiation condition, the scattered wave has the form of outgoing radiation and is a regular solution of Eq. 28 everywhere but at $\vec{r} = \vec{r}_m$, where a singularity is present.

The properties of a single scatterer are assumed known, so that a rule is available relating the scattered wave and the exciting field $\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N)$ acting on the m^{th} scatterer to produce scattering. This rule defines a linear scattering operator $T(\vec{q}_m)$ by the relation

$$\psi^s(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N) = T(\vec{q}_m)\,\psi^E(\vec{r},\omega|q_m|\vec{q}_1,\ldots,\vec{q}_N) \; . \tag{31}$$

The exciting field is assumed to be a regular solution of the Helmholtz equation, Eq. 28, in the neighborhood of $\vec{r}_m$ in order that the single scatterer computation be applicable.

By the principle of superposition, the total field $\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_N)$ may now be written as

$$\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_N) = \psi^i(\vec{r},\omega) + \sum_{m=1}^{N} \psi^S(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N) \ , \tag{32}$$

or using Eq. 31,

$$\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_N) = \psi^i(\vec{r},\omega) + \sum_{m=1}^{N} T(\vec{q}_m)\,\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N) \ . \tag{33}$$

Now the exciting field acting on the m^{th} scatterer is just the total field minus the wave scattered by the m^{th} scatterer, i.e.,

$$\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N) = \psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_N) - \psi^S(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N) \ . \tag{34}$$

Using Eqs. 31 and 33, this last relation can be written as

$$\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N) = \psi^i(\vec{r},\omega) + \sum_{\substack{m'=1 \\ (m'\neq m)}}^{N} T(\vec{q}_{m'})\,\psi^E(\vec{r},\omega|\vec{q}_{m'}|\vec{q}_1,\ldots,\vec{q}_N) \ . \tag{35}$$

Eq. 35 represents a set of N algebraic equations for the N function $\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N)$, $1\leq m\leq N$. Once these N functions, representing the exciting field on each scatterer, are known, they can be substituted into Eq. 33 to give the total field.

These relations account completely for the effect on each scatterer due to the presence of other scatterers. In terms of multiple orders of scattering approach where primary (or first order) scattering is due to the incident wave alone, secondary (or second order) scattering represents one rescattering of the primary waves, and so on, all orders are included in Eqs 33 and 35.

To summarize, the self-consistent field equations of multiple scattering are given by

$$\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_N)=\psi^i(\vec{r},\omega)+\sum_{m=1}^{N} T(\vec{q}_m)\,\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N)\ , \qquad (36)$$

$$\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N)=\psi^i(\vec{r},\omega)+\sum_{\substack{m'=1\\(m'\neq m)}}^{N} T(\vec{q}_{m'})\,\psi^E(\vec{r},\omega|\vec{q}_{m'}|\vec{q}_1,\ldots,\vec{q}_N). \qquad (37)$$

In order to gain some insight into Eqs. 36 and 37 let us consider the case when N=2. Eq. 36 becomes

$$\psi(\vec{r},\omega|\vec{q}_1,\vec{q}_2) = \psi^i(\vec{r},\omega)+T(\vec{q}_1)\,\psi^E(\vec{r},\omega|\vec{q}_1|\vec{q}_1,\vec{q}_2)$$

$$+T(\vec{q}_2)\,\psi^E(\vec{r},\omega|\vec{q}_2|\vec{q}_1,\vec{q}_2)\ , \qquad (38)$$

and Eq. 37 becomes

$$\psi^E(\vec{r},\omega|\vec{q}_1|\vec{q}_1,\vec{q}_2) = \psi^i(\vec{r},\omega) + T(\vec{q}_2)\,\psi^E(\vec{r},\omega|\vec{q}_2|\vec{q}_1,\vec{q}_2)\ , \qquad (39)$$

$$\psi^E(\vec{r},\omega|\vec{q}_2|\vec{q}_1,\vec{q}_2) = \psi^i(\vec{r},\omega) + T(\vec{q}_1)\,\psi^E(\vec{r},\omega|\vec{q}_1|\vec{q}_1,\vec{q}_2)\ . \qquad (40)$$

Solving Eqs. 39 and 40 for $\psi^E(\vec{r},\omega|\vec{q}_1|\vec{q}_1,\vec{q}_2)$ and $\psi^E(\vec{r},\omega|\vec{q}_2|\vec{q}_1,\vec{q}_2)$, yields

$$\psi^E(\vec{r},\omega|\vec{q}_1|\vec{q}_1,\vec{q}_2) = \frac{\psi^i(\vec{r},\omega) + T(\vec{q}_2)\,\psi^i(\vec{r},\omega)}{1 - T(\vec{q}_2)\,T(\vec{q}_1)}\ , \qquad (41)$$

$$\psi^E(\vec{r},\omega|\vec{q}_2|\vec{q}_1,\vec{q}_2) = \frac{\psi^i(\vec{r},\omega) + T(\vec{q}_1)\,\psi^i(\vec{r},\omega)}{1 - T(\vec{q}_1)\,T(\vec{q}_2)}\ . \qquad (42)$$

The total field is obtained by substituting these values, Eqs. 41 and 42, into Eq. 38.

$$\psi(\vec{r},\omega|\vec{q}_1,\vec{q}_2) = \psi^i(\vec{r},\omega) + T(\vec{q}_1)\left\{\frac{\psi^i(\vec{r},\omega)+T(\vec{q}_2)\,\psi^i(\vec{r},\omega)}{1 - T(\vec{q}_2)\,T(\vec{q}_1)}\right\}$$

$$+ T(\vec{q}_2)\left\{\frac{\psi^i(\vec{r},\omega) + T(\vec{q}_1)\,\psi^i(\vec{r},\omega)}{1 - T(\vec{q}_1)\,T(\vec{q}_2)}\right\}. \qquad (43)$$

To show explicitly that Eq. 43 includes all orders of multiple scattering, we expand the denominator of the last two terms on the right side of Eq. 43 in a power series. Assuming that $T(\vec{q}_1)\ T(\vec{q}_2) = T(\vec{q}_2)\ T(\vec{q}_1)$, we then have for both denominators

$$\left[1 - T(\vec{q}_2)\ T(\vec{q}_1)\right]^{-1} = 1 + T(\vec{q}_2)\ T(\vec{q}_1) + T(\vec{q}_2)T(\vec{q}_1)T(\vec{q}_2)T(\vec{q}_1) + \ldots , \tag{44}$$

provided that $|T(\vec{q}_2)\ T(\vec{q}_1)| < 1$. Thus Eq. 43 becomes

$$\begin{aligned}\psi(\vec{r},\omega|\vec{q}_1,\vec{q}_2) = {} & \psi^i(\vec{r},\omega) + T(\vec{q}_1)\,\psi^i(\vec{r},\omega) + T(\vec{q}_2)\,\psi^i(\vec{r},\omega) \\ & + T(\vec{q}_1)\left[T(\vec{q}_2)\,\psi^i(\vec{r},\omega)\right] + T(\vec{q}_2)\left[T(\vec{q}_1)\,\psi^i(\vec{r},\omega)\right] \\ & + T(\vec{q}_1)\left\{T(\vec{q}_2)\left[T(\vec{q}_1)\,\psi^i(\vec{r},\omega)\right]\right\} \\ & + T(\vec{q}_2)\left\{T(\vec{q}_1)\left[T(\vec{q}_2)\,\psi^i(\vec{r},\omega)\right]\right\} + \ldots \end{aligned} \tag{45}$$

The term $\psi^i(\vec{r},\omega)$ is the value of the incident wave at the point $\vec{r}$ after having undergone no scatterings; $T(\vec{q}_1)\,\psi^i(\vec{r},\omega)$ is the value of the incident wave at the point $\vec{r}$ after having been scattered once by the scatterer located at $\vec{r}_1$; $T(\vec{q}_2)\,\psi^i(r,\omega)$ is the value of the incident wave at the point $\vec{r}$ after having been scattered once by the scatterer located at $\vec{r}_2$; $T(\vec{q}_1) \times \left[T(\vec{q}_2)\,\psi^i(r,\omega)\right]$ is the value of the incident wave at the point $\vec{r}$

after having been scattered twice - first by the scatterer located at r_2 and then by the scatterer located at $\vec{r}_1$; etc.

It is important to note that the explicit resolution of the total field into multiple orders of scattering, Eq. 45, is valid only if the interaction between the scatterers is sufficiently weak so that the condition $|T(\vec{q}_1)\ T(\vec{q}_2)| < 1$ holds. If the wave about each scatterer is profoundly influenced by the presence of the other scatterer, i.e., if there is a strong interaction between the scatterers, then this condition may not be met and the explicit resolution into multiple orders of scattering loses its meaning. In this case the more rigorous Eq. 43 must be used. While this equation accounts for all orders of multiple scattering, it does so implicitly.

If the scattering is extremely weak, i.e., if $|T(\vec{q}_1)\ T(\vec{q}_2)| << 1$, then all second and higher order scattering terms can be neglected in Eq. 45 and we have

$$\psi(\vec{r},\omega|\vec{q}_1,\vec{q}_2)=\psi^i(\vec{r},\omega)+T(\vec{q}_1)\,\psi^i(\vec{r},\omega)+T(\vec{q}_2)\,\psi^i(\vec{r},\omega)\ . \qquad (46)$$

This is called the Born approximation.

E. AVERAGED FIELDS

Because of the complicated nature of Eqs. 36 and 37, it does not appear feasible to attempt to invert Eqs. 37 to obtain explicit expression for the exciting fields. Instead, the

equations will be averaged as they stand. First let us average Eq. 36 for the total field:

$$\langle\psi(\vec{r},\omega)\rangle = \int\cdots\int \psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_N)p(\vec{q}_1,\ldots,\vec{q}_N)d\vec{q}_1\ldots d\vec{q}_N$$

$$= \psi^i(\vec{r},\omega) + \sum_{m=1}^{N} \int\cdots\int T(\vec{q}_m)\,\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N)p(\vec{q}_1,\ldots,\vec{q}_N)d\vec{q}_1\ldots d\vec{q}_N$$

$$= \psi^i(\vec{r},\omega) + \sum_{m=1}^{N} \int\cdots\int T(\vec{q}_m)\,\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N)$$

$$\times\ p(\vec{q}_m)p(\vec{q}_1,\ldots,\vec{q}_{m-1},\vec{q}_{m+1}\ldots,\vec{q}_N|\vec{q}_m)\ d\vec{q}_1\ldots d\vec{q}_N$$

$$= \psi^i(\vec{r},\omega) + \sum_{m=1}^{N} \int T(\vec{q}_m)p(\vec{q}_m)d\vec{q}_m \int\cdots\int d\vec{q}_1\ldots d\vec{q}_{m-1}d\vec{q}_{m+1}\ldots d\vec{q}_N$$

$$\times\ \psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N)p(\vec{q}_1,\ldots,\vec{q}_{m-1},\vec{q}_{m+1},\ldots,\vec{q}_N|\vec{q}_m)$$

$$= \psi^i(\vec{r},\omega) + \sum_{m=1}^{N} \int T(\vec{q}_m)p(\vec{q}_m)\langle\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_m)\rangle\ d\vec{q}_m\ . \qquad (47)$$

Here use has been made of Eqs. 6, 16 and 17. Thus to evaluate the ensemble average of the total field, $\langle\psi(\vec{r},\omega)\rangle$, we need to determine the partial averages $\langle\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_m)\rangle$ for $1 \leq m \leq N$. The quantity $\langle\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_m)\rangle$ is the ensemble average of the exciting field acting on the m^{th} scatterer when the m^{th} scatterer is held

fixed. Let us compute the first of these quantities, viz., $\langle \psi^E(\vec{r},\omega|\vec{q}_1|\vec{q}_1)\rangle$.

From Eq. 37, we have

$$\psi^E(\vec{r},\omega|\vec{q}_1|\vec{q}_1,\ldots,\vec{q}_N)=\psi^i(\vec{r},\omega)+\sum_{m=2}^{N} T(\vec{q}_m)\,\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N).$$

Consequently,

$$\langle \psi^E(\vec{r},\omega|\vec{q}_1|\vec{q}_1)\rangle = \int\cdots\int \psi^E(\vec{r},\omega|\vec{q}_1|\vec{q}_1,\ldots,\vec{q}_N)\,p(\vec{q}_2,\ldots,\vec{q}_N|\vec{q}_1)\,d\vec{q}_2\ldots d\vec{q}_N$$

$$= \psi^i(\vec{r},\omega)+\sum_{m=2}^{N}\int\cdots\int T(\vec{q}_m)\,\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N)$$

$$\times\, p(\vec{q}_2,\ldots,\vec{q}_N|\vec{q}_1)\,d\vec{q}_2\ldots d\vec{q}_N \quad . \tag{48}$$

A slight generalization of Eq. 13 yields

$$p(\vec{q}_2,\ldots,\vec{q}_N|\vec{q}_1)=p(\vec{q}_m|\vec{q}_1)\,p(\vec{q}_2,\ldots,\vec{q}_{m-1},\vec{q}_{m+1},\ldots,\vec{q}_N|\vec{q}_1,\vec{q}_m) \quad . \tag{49}$$

Thus Eq. 48 can be written as

$$\langle \psi^E(\vec{r},\omega|\vec{q}_1|\vec{q}_1)\rangle = \psi^i(\vec{r},\omega) + \sum_{m=2}^{N} \int T(\vec{q}_m) p(\vec{q}_m|\vec{q}_1) d\vec{q}_m$$

$$\times \int \ldots \int \psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N) p(\vec{q}_2,\ldots,\vec{q}_{m-1},\vec{q}_{m+1},\ldots,\vec{q}_N|\vec{q}_1,\vec{q}_m)$$

$$\times d\vec{q}_2 \ldots d\vec{q}_{m-1} d\vec{q}_{m+1} \ldots d\vec{q}_N$$

$$\langle \psi^E(\vec{r},\omega|\vec{q}_1|\vec{q}_1)\rangle = \psi^i(r,\omega) + \sum_{m=2}^{N} \int T(\vec{q}_m) p(\vec{q}_m \mid \vec{q}_1)$$

$$\times \langle \psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\vec{q}_m)\rangle d\vec{q}_m \quad . \qquad (50)$$

Thus we get the very unsatisfactory result that the average of the exciting field with one scatterer fixed is given in terms of the average of the exciting field with two scatterers fixed. Further calculation shows that higher partially averaged exciting fields will involve the integral of the exciting field with one additional scatterer held fixed. This lack of completeness is the basic difficulty encountered in the implicit approach to multiple scattering. Lax[13] has suggested breaking off this hierarchy of equations at some stage by arbitrarily replacing the exciting field in an integrand by the corresponding field with one less scatterer held fixed. The resulting equation is then solved, and each of the preceding equations solved by quadrature.

The system of exact equations governing multiple scattering is now complete. There are three known methods for obtaining a solution to these equations. First, one can in principle obtain an exact solution by solving the implicit initial representation, Eq. 37, for the N quantities $\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N)$. This method was illustrated by the example for the case when N = 2 with the solutions being given by Eqs. 41 and 42. The solutions for $\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N)$, obtained by solving the system of equations, Eq. 37, can then be substituted into Eq. 47 to give an exact expression for the ensemble average of the total field. However, this scheme must be discarded because the labor is prohibitive. Second, one might iterate the system (eq. 37), replacing the exciting field in the summation of one equation by the right-hand side of the following equation, and continuing to replace the exciting field wherever it occurs, ultimately obtaining an infinite series representation for each of the exciting fields which involves only operations on $\psi^i(\vec{r},\omega)$. This result expresses the exciting field in terms of multiple orders of scattering and leads to the system of equations

$$\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N)=\psi^i(\vec{r},\omega)+\sum_{\substack{m'=1\\(m'\neq m)}}^{N} T(\vec{q}_{m'})\,\psi^i(\vec{r},\omega)$$

$$+\sum_{\substack{m''=1\\(m''\neq m)}}^{N} T(\vec{q}_{m''})\sum_{\substack{m'=1\\(m'\neq m)}}^{N} T(\vec{q}_{m'})\,\psi^i(\vec{r},\omega)+\ldots \quad . \tag{51}$$

This result is then substituted into Eq. 47 to obtain the average for the total field. Such series rapidly become cumbersome to treat exactly when more than one or two terms are considered, so that this scheme is only feasible when multiple scattering effects are very weak. Finally, one can average Eq. 37 as it stands. This is the technique that was used in this section and it was seen to lead to a complicated hierarchy of equations. Even if Lax's suggestion is used, the labor may be prohibitive. Thus, if we are to arrive at a solution, some kind of approximation must be used. One such approximation will be discussed in the next section.

F. APPROXIMATE EQUATIONS

We begin by expressing the exciting field $\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N)$ incident on the m^{th} scatterer in terms of the total field $\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_{m-1},\vec{q}_{m+1},\ldots,\vec{q}_N)$ that would exist in the neighborhood of the m^{th} scatterer if the m^{th} scatterer were not present. That is, we first remove the m^{th} scatterer from our configuration. Then let $\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_{m-1},\vec{q}_{m+1},\ldots,\vec{q}_N)$ be the total field in the neighborhood of the position where the

m^{th} scatterer was originally before removal. We now insert the m^{th} scatterer back into its original position. Due to its presence, the wave $\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_{m-1},\vec{q}_{m+1},\ldots,\vec{q}_N)$, which existed in the vacant neighborhood of the m^{th} scatterer, is scattered by the m^{th} scatterer and then rescattered by the other scatterers, consquently, changing the value of the wave incident on the m^{th} scatterer from $\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_{m-1},\vec{q}_{m+1},\ldots,\vec{q}_N)$ to $\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N)$. We can express this as

$$\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N)=\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_{m-1},\vec{q}_{m+1},\ldots,\vec{q}_N) + \sum_{\substack{k=1\\(k\neq m)}}^{N} \chi^S(\vec{r},\omega|\vec{q}_k|\vec{q}_1,\ldots,\vec{q}_N) \ , \tag{52}$$

where $\chi^S(\vec{r},\omega|\vec{q}_k|\vec{q}_1,\ldots,\vec{q}_N)$ is the wave scattered by the k^{th} scatterer. Let $\chi^E(\vec{r},\omega|\vec{q}_k|\vec{q}_1,\ldots,\vec{q}_N)$ be the wave incident on the k^{th} scatterer to produce the scattered wave $\chi^S(\vec{r},\omega|\vec{q}_k|\vec{q}_1,\ldots,\vec{q}_N)$, i.e., let

$$\chi^S(\vec{r},\omega|\vec{q}_k|\vec{q}_1,\ldots,\vec{q}_N)=T(\vec{q}_k)\chi^E(\vec{r},\omega|\vec{q}_k|\vec{q}_1,\ldots,\vec{q}_N) \ . \tag{53}$$

From the above discussion we can write

$$\chi^E(\vec{r},\omega|\vec{q}_k|\vec{q}_1,\ldots,\vec{q}_N)=T(\vec{q}_m)\,\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_{m-1},\vec{q}_{m+1},\ldots,\vec{q}_N) + \sum_{\substack{j=1\\(j\neq m,k)}}^{N} T(\vec{q}_j)\chi^E(\vec{r},\omega|\vec{q}_j|\vec{q}_1,\ldots,\vec{q}_N) \ . \tag{54}$$

We now apply the same iteration technique to Eq. 54 that we used to obtain Eq. 51 from Eq. 37. This yields

$$\chi^E(\vec{r},\omega|\vec{q}_k|\vec{q}_1,\ldots,\vec{q}_N)=T(\vec{q}_m)\,\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_{m-1},\vec{q}_{m+1},\ldots,\vec{q}_N)$$

$$+\sum_{\substack{j=1\\(j\neq m,k)}}^{N} T(\vec{q}_j)T(\vec{q}_m)\,\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_{m-1},\vec{q}_{m+1},\ldots,\vec{q}_N)$$

$$+\ldots \quad . \tag{55}$$

If we now combine Eqs. 52, 53 and 55, we have

$$\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N)=\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_{m-1},\vec{q}_{m+1},\ldots,\vec{q}_N)$$

$$+\sum_{k\neq m} T(\vec{q}_k)\Big[1+\sum_{j\neq k,m} T(\vec{q}_j)+\sum_{j\neq k,m} T(\vec{q}_j)\sum_{n\neq j,m} T(\vec{q}_n)$$

$$+\sum_{j\neq k,m} T(\vec{q}_j)\sum_{n\neq j,m} T(\vec{q}_n)\sum_{p\neq n,m} T(\vec{q}_p)+\ldots\Big]$$

$$\times\, T(\vec{q}_m)\,\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_{m-1},\vec{q}_{m+1},\ldots,\vec{q}_N) \quad . \tag{56}$$

It should be stressed that this representation, while not in closed form, is exact.

Now a great simplification in the equations of multiple scattering would result if we could make the approximation

$$\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N) \approx \psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_{m-1},\vec{q}_{m+1},\ldots,\vec{q}_N) , \qquad (57)$$

obtained by neglecting all but the first term on the right side of Eq. 56. Unfortunately, there are no known general conditions which tell us when these terms can be neglected. However, for the case of isotropic scattering an estimate can be made and this will be discussed in Section G.

Using the approximation given by Eq. 57, that the exciting field on a scatterer may be approximated by the total field existent in the neighborhood if the scatterer were not there, we can write Eq. 36 as

$$\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_N) = \psi^i(\vec{r},\omega) + \sum_{m=1}^{N} T(\vec{q}_m) \times \psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_{m-1},\vec{q}_{m+1},\ldots,\vec{q}_N) . \qquad (58)$$

Now let us assume that the locations and radii of the scatterers are statistically independent so that Eq. 18 holds. Then taking the ensemble average of Eq. 58 yields

$$\langle\psi(\vec{r},\omega)\rangle = \psi^i(\vec{r},\omega) + \sum_{m=1}^{N} \int p(\vec{q}_m) T(\vec{q}_m) d\vec{q}_m \int\cdots\int d\vec{q}_1\cdots d\vec{q}_{m-1} d\vec{q}_{m+1}\cdots d\vec{q}_N \times p(\vec{q}_1)\cdots p(\vec{q}_{m-1}) p(\vec{q}_{m+1})\cdots p(\vec{q}_N)\, \psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_{m-1},\vec{q}_{m+1},\ldots,\vec{q}_N) . \qquad (59)$$

The (N-1) - fold integral in Eq. 59 is independent of $\vec{q}_m$ and, in fact, is precisely the average total field for the same problem with one less scatterer involved. As the number of scatterers increases, this quantity must differ from the total field with N scatterers by terms of order 1/N. Thus, assuming a large number of scatterers, we make the approximation

$$\langle\psi(\vec{r},\omega)\rangle = \int\ldots\int d\vec{q}_1\ldots d\vec{q}_N\,\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_N)p(\vec{q}_1)\ldots p(\vec{q}_N)$$

$$=\int\ldots\int d\vec{q}_1\ldots d\vec{q}_{m-1}d\vec{q}_{m+1}\ldots d\vec{q}_N\,\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_{m-1},\vec{q}_{m+1},\ldots,\vec{q}_N)$$

$$\times\, p(\vec{q}_1)\ldots p(\vec{q}_{m-1})p(\vec{q}_{m+1})\ldots p(\vec{q}_N)\,. \quad (60)$$

Consequently, Eq. 59 becomes

$$\langle\psi(\vec{r},\omega)\rangle = \psi^i(\vec{r},\omega) + \sum_{m=1}^{N} \int T(\vec{q}_m)p(\vec{q}_m)\langle\psi(\vec{r},\omega)\rangle\, d\vec{q}_m\,. \quad (61)$$

Now using Eq. 5 and assuming that

$$p(\vec{q}_1) = p(\vec{q}_2) = \ldots = p(\vec{q}_N)\,,$$

$$n(\vec{q}_1) = n(\vec{q}_2) = \ldots = n(\vec{q}_N)\,, \quad (62)$$

We can write Eq. 61 as

$$\langle\psi(\vec{r},\omega)\rangle = \psi^i(\vec{r},\omega) + \int T(\vec{q}\,')\,n(\vec{q}\,')\langle\psi(\vec{r},\omega)\rangle d\vec{q}\,'\,. \quad (63)$$

This integral equation will be the starting point for our investigation of the multiple scattering from fish schools.

G. ISOTROPIC SCATTERING

For the case of isotropic scattering, the operator $T(\vec{q}_m)$ is defined by

$$T(\vec{q}_m)\,\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N)=g(\vec{q}_m,\omega)\,\psi^E(\vec{r}_m,\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N)\frac{e^{i\kappa_o|\vec{r}-\vec{r}_m|}}{|\vec{r}-\vec{r}_m|}\ . \tag{64}$$

Thus, the integrand on the right side of Eq. 63 becomes

$$\begin{aligned}
T(\vec{q}\,')\langle\psi(\vec{r},\omega)\rangle &= T(\vec{q}\,')\int\cdots\int\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_N)p(\vec{q}_1,\ldots,\vec{q}_N)d\vec{q}_1\ldots d\vec{q}_N \\
&= \int\cdots\int p(\vec{q}_1,\ldots,\vec{q}_N)T(\vec{q}\,')\,\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_N)d\vec{q}_1\ldots d\vec{q}_N \\
&= \int\cdots\int p(\vec{q}_1,\ldots,\vec{q}_N)g(\vec{q}\,',\omega)\,\psi(\vec{r}\,',\omega|\vec{q}_1,\ldots,\vec{q}_N)\,\frac{e^{i\kappa_o|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\,d\vec{q}_1\ldots d\vec{q}_N \\
&= g(\vec{q}\,',\omega)\,\frac{e^{i\kappa_o|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\,\langle\psi(\vec{r}\,',\omega)\rangle\ .
\end{aligned} \tag{65}$$

The operator $T(\vec{q}\,')$ commutes with the N-fold integration since $\vec{q}\,'$ is independent of $\vec{q}_1,\ldots,\vec{q}_N$ and, so far as the N-fold integration is concerned, the variables $\vec{q}_1,\ldots,\vec{q}_N$ are simply dummy variables.

Consequently, using Eq. 65, our integral equation for multiple scattering, Eq. 63, becomes for the case of isotropic scattering

$$\langle\psi(\vec{r},\omega)\rangle = \psi^{i}(\vec{r},\omega)+\int\int n(\vec{r}^{\,\prime},a^{\prime})g(\vec{r}^{\,\prime},a^{\prime},\omega)\langle\psi(\vec{r}^{\,\prime},\omega)\rangle \frac{e^{i\kappa_o|\vec{r}-\vec{r}^{\,\prime}|}}{|\vec{r}-\vec{r}^{\,\prime}|}\, d\vec{r}^{\,\prime} da^{\prime}. \quad (66)$$

Defining

$$G(\vec{r}^{\,\prime},\omega)=\int n(\vec{r}^{\,\prime},a^{\prime})g(\vec{r}^{\,\prime},a^{\prime},\omega)da^{\prime}, \quad (67)$$

Eq. 66 takes the final form

$$\langle\psi(\vec{r},\omega)\rangle = \psi^{i}(\vec{r},\omega)+\int G(\vec{r}^{\,\prime},\omega)\langle\psi(\vec{r}^{\,\prime},\omega)\rangle \frac{e^{i\kappa_o|\vec{r}-\vec{r}^{\,\prime}|}}{|\vec{r}-\vec{r}^{\,\prime}|}\, d\vec{r}^{\,\prime}. \quad (68)$$

To see the physical significance of this equation, let us apply the operator

$$\nabla^2 + \kappa_o^2 \quad (69)$$

to both sides of the integral equation, Eq. 68, remembering that

$$(\nabla^2 + \kappa_o^2)\, \frac{e^{i\kappa_o|\vec{r}-\vec{r}^{\,\prime}|}}{|\vec{r} - \vec{r}^{\,\prime}|} = -4\pi\delta(\vec{r}-\vec{r}^{\,\prime}) , \quad (70)$$

where $\delta(\vec{r}-\vec{r}^{\,\prime})$ is the three dimensional Dirac δ-function defined by the equation

$$\int_V f(\vec{r}')\delta(\vec{r}-\vec{r}')d\vec{r}' = f(\vec{r}) \text{ , if } \vec{r}\in V$$

$$= o, \text{ if } \vec{r} \notin V . \tag{71}$$

We then obtain

$$\nabla^2\langle\psi(\vec{r},\omega)\rangle + k^2(\vec{r},\omega)\langle\psi(\vec{r},\omega)\rangle = o , \tag{72}$$

where

$$k^2(\vec{r},\omega) = \kappa_o^2 + 4\pi\, G(\vec{r},\omega) . \tag{73}$$

In general, $g(\vec{r},a,\omega)$ will be complex, so that $k(\vec{r},\omega)$ will be complex. Thus, we see that $\langle\psi(\vec{r},\omega)\rangle$ satisfies the Helmholtz equation having a complex wave number that depends, in general, on position and frequency. Consequently, the incident wave and the scattered waves from all the scatterers interfere, on the average, to form a new wave traveling at a different phase velocity and undergoing attenuation. This wave will display the reflection and refraction aspects of coherent scattering at surfaces of discontinuity.

The problem of finding the average value of the wave function has been essentially reduced to solving a boundary value problem in the wave equation. The boundary conditions are implied in the integral equation itself and depend on the function $G(\vec{r},\omega)$. If $G(\vec{r},\omega)$ is everywhere continuous and approaches a constant value or zero at infinity, then the

boundary conditions are that $\langle\psi(\vec{r},\omega)\rangle - \psi^i(\vec{r},\omega)$ be everywhere continuous and have a continuous gradient and at infinity represent outward traveling waves. In another important case, that in which $G(\vec{r},\omega)$ is sectionally continuous, the boundary conditions are that $\langle\psi(\vec{r},\omega)\rangle - \psi^i(\vec{r},\omega)$ be everywhere continuous and have a continuous normal derivative across a surface of discontinuity of $G(\vec{r},\omega)$ and at infinity represent outward traveling waves, provided again that $G(\vec{r},\omega)$ approaches a constant value or zero at infinity. In both cases, of course, $\langle\psi(\vec{r},\omega)\rangle$ must approach $\psi^i(\vec{r},\omega)$ as $G(\vec{r},\omega)$ becomes zero everywhere.

We note also that the integral equation, Eq. 68, can in principle be solved directly. One method of solution is the Liouville-Neumann method of successive substitutions or iteration method. It consists in repeatedly substituting the expression for $\langle\psi(\vec{r},\omega)\rangle$ as given by the right side of the integral equation for $\langle\psi(\vec{r}\,',\omega)\rangle$ under the integral sign, thus yielding in our case the infinite series

$$\langle\psi(\vec{r},\omega)\rangle = \sum_{m=o}^{\infty} \psi_m(\vec{r},\omega) \; , \qquad (74)$$

where

$$\psi_m(\vec{r},\omega) = \int G(\vec{r}\,',\omega)\, \psi_{m-1}(\vec{r}\,',\omega)\, \frac{e^{i\kappa_o|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\, d\vec{r}\,' , (m \neq o) \; , \qquad (75)$$

which, if it converges uniformly, is the desired solution. In the above notation $\psi_o(\vec{r},\omega) = \psi^i(\vec{r},\omega)$. If the scattering is very weak, it may be possible to neglect all terms for which m > 2. In this case Eq. 74 takes the form

$$\langle\psi(\vec{r},\omega)\rangle = \psi^i(\vec{r},\omega) + \int G(\vec{r}',\omega)\ \psi^i(\vec{r}',\omega)\ \frac{e^{i\kappa_o|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\ d\vec{r}' . \qquad (76)$$

This approximation is known as the Born approximation.

In order to obtain the approximate integral equation, Eq. 63, or for the case of isotropic scattering, Eq. 68, we made the assumption that

$$\psi^E(\vec{r},\omega|\vec{q}_m|\vec{q}_1,\ldots,\vec{q}_N) \approx \psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_{m-1},\vec{q}_{m+1},\ldots,\vec{q}_N) .$$

Let us now examine the conditions under which this approximation is valid for the case of isotropic scattering. The argument while plausible is by no means rigorous. Consider the first term, say φ_o, of the series in Eq. 56, i.e., let

$$\varphi_o(\vec{r}) = \sum_{k\neq m} T(\vec{q}_k)\ T(\vec{q}_m)\,\psi(\vec{r},\omega|\vec{q}_1,\ldots,\vec{q}_{m-1},\vec{q}_{m+1},\ldots,\vec{q}_N). \qquad (77)$$

Each term in the summation represents rescattering from one of the obstacles of an outgoing scattered wave from $\vec{r}_m$. Furthermore, since the exciting field is only required in the immediate neighborhood of the scatterer, in question, in this case the m^{th} according to Eq. 56, we need only estimate $\varphi_o(\vec{r})$ in this

neighborhood; for simplicity consider $\varphi_o(\vec{r}_m)$. Surfaces of equal phase of the outgoing wave will be spherical so that the phase of rescattered waves returning to $\vec{r}_m$ retard according to the round trip distance $2\,|\vec{r}_k - \vec{r}_m|$ from $\vec{r}_m$ to the individual scatterer. A set of concentric half-period zones may be constructed about $\vec{r}_m$, each zone defined by the requirement that its rescattered waves are no more than half a period out of phase at $\vec{r}_m$. These zones are illustrated in Fig. 1.

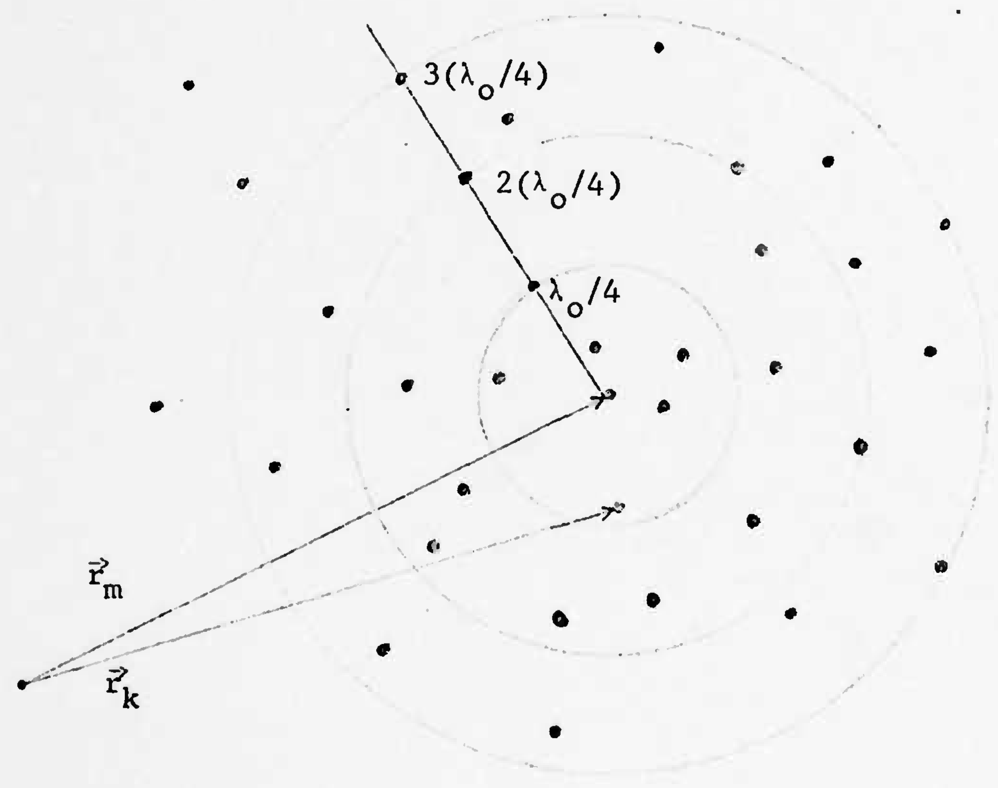

Fig. 1

Suppose there is a uniform random distribution of density n and further that the scattering amplitude $g(\vec{r}_k, a_k, \omega)$, defined by Eq. 64, is independent of the position $\vec{r}_k$ of the k^{th} scatterer and that $g(a, \omega)$ is identical for all scatterers. Then using the fact that the scattering operator T is defined by Eq. 64 for the case of isotropic scattering and using the abbreviation $\psi(\vec{r}_m) = \psi(\vec{r}_m, \omega | \vec{q}_1, \ldots, \vec{q}_{m-1}, \vec{q}_{m+1}, \ldots, \vec{q}_N)$, the expression for $\varphi_o(\vec{r}_m)$ becomes

$$\varphi_o(\vec{r}_m) = \sum_{k \neq m} g^2 \psi(\vec{r}_m) \frac{e^{2i\kappa_o |\vec{r}_k - \vec{r}_m|}}{|\vec{r}_k - \vec{r}_m|^2} . \tag{78}$$

The contributions from succeeding zones are the same order of magnitude but alternating in sign, the square law increase of number of scatterers with distance being precisely compensated by the inverse behavior of both scattered and rescattered waves. One concludes that summation over the first zone should suffice to give an order of magnitude estimate of $\varphi_o(\vec{r}_m)$, and further approximating the sum by an integral we obtain

$$\varphi_o(\vec{r}_m) \approx g^2 \psi(\vec{r}_m) \sum_{k \neq m}^{\text{1st zone}} \frac{e^{2i\kappa_o |\vec{r}_k - \vec{r}_m|}}{|\vec{r}_k - \vec{r}_m|^2}$$

$$\approx n\, g^2 \psi(\vec{r}_m) \int_{\text{1st zone}} \frac{e^{2i\kappa_o |\vec{r}\,' - \vec{r}_m|}}{|\vec{r}\,' - \vec{r}_m|^2}\, d\vec{r}\,'$$

$$\approx \left(\frac{4\pi n i}{\kappa_o}\right) g^2\ \psi(\vec{r}_m). \tag{79}$$

Thus $\varphi_o(\vec{r}_m)$ may be neglected in comparison with $\psi(\vec{r}_m, \omega | \vec{q}_1, \ldots, \vec{q}_{m-1}, \vec{q}_{m+1}, \ldots, \vec{q}_N)$ in Eq. 56 provided

$$\left| \frac{4\pi n i}{\kappa_o}\ g^2 \right| = \frac{4\pi n}{\kappa_o} |g|^2 << 1\ . \tag{80}$$

Since $4\pi |g|^2$ can be shown to be just the scattering cross section σ of a single scatterer, the above criterion simplifies to

$$\frac{n\sigma}{\kappa_o} << 1. \tag{81}$$

The above estimates have only been concerned with the primary order of the multiple orders of scattering correction to the exciting field, i.e., the first term of the infinite

series appearing on the right side of Eq. 56. One can then infer that the estimate remains valid when the entire series is considered[6].

Part II. SCATTERING FROM FISH SCHOOLS

A. INTRODUCTION

The general theory of the multiple scattering of a pulse by a random collection of point scatterers that was developed in Part I is now applied to the case where the scatterer is a bladder fish. The reason for confining our attention to bladder fish will be explained later in this section. In general, we may distinguish between "scattering groups" and "scattering layers". By a "scattering group" we shall mean a collection of fish such that the horizontal dimensions are at most only a few times larger than the vertical dimension. By a "scattering layer" we shall mean a collection of fish such that the horizontal dimensions are a great many times larger than the vertical dimension. A "scattering layer" can be represented mathematically by a volume bounded by two horizontal planes which extend to infinity. In general, the planes will have an uneven surface. As far as this report is concerned, a school of fish will mean a "scattering group" while the only "scattering layer" that we shall be interested in is the Deep Scattering Layer (DSL). Even though we mentioned in the Introduction to this report that the DSL does not appear to be a likely source of false contact for operation in the BB mode of propagation since it would be detected simultaneously on more than one beam, we are considering it

briefly in this report because of its importance in producing reverberation and because it is an immediate consequence of our theory.

Large areas of the deep ocean have a virtually continuous DSL. These are formed of various sorts of marine organism, but especially of bathypelagic fish. Out of the many studies attention should be drawn to those that have looked at the frequency structure, and by using wide-band underwater explosion sources have demonstrated resonances.[14,15,16] These results are in fact the strongest evidence that resonant swim bladders do play an important part in fish acoustics. Extensive scattering layers can also occur in shallow coastal waters and are frequently associated with summer thermoclines.

Tightly packed schools of fish can occur in both the deep ocean and in shallow water, and they come in a great variety of shapes, sizes and concentrations. The records obtained by Voglis and Cook[21] show some of this variety for schools of pilchard. The school sizes which have been seen with the Voglis equipment vary from a few yards across to those with a maximum dimension of about 40 yards (the soundings were taken looking down obliquely from above). The shapes vary from the near circular to the line, a surprising number have U-shape. The differences in the apparent strength of return, as measured

by Voglis, must not be interpreted solely in terms of mean fish concentration, it may also depend on the roughness or diffuseness of the school boundary.

While scattering of sound by fish is often observed, the possibility of significant attenuation due to fish has not been discussed in the literature. The status of the subject of low frequency attenuation is that in deep water its high value is still unexplained, and in shallow water Weston and his colleaques have made direct observations proving that a large part of the long range attenuation is sometimes due to fish. While this report is only concerned with calculating the first order statistics of the signal scattered by a fish school, future work will deal with the second order statistics of the scattered signal. Consequently, once the second order statistics are known, the question of whether there is significant attentuation due to fish schools will be answered.

Now let us examine the theoretical and experimental work which has been done on sound scattering from fish. For the deep scattering layer, we have already mentioned that sound scattering measurements have been made, e.g., by Hersey, et. al.,[14] Marshall, et. al.,[15] and Andreeva[16]. The scattering of a pulse from the DSL, neglecting multiple scattering, has been theoretically calculated by Glotov[17],

Glotov, et. al.[18], and Kuryanov[19]. There seems to have been no work done which describes the multiple scattering of either an incident monochromatic wave or an incident pulse by the DSL.

No theoretical work of any kind appears to have been done on sound scattering from fish schools and only a very limited amount of experimental work. Only three papers could be found which reported measured values: Wickham, et. al.[20], Voglis, et. al.[21], and McCartney, et. al.[22] Of these three papers, only the first two and another paper by McFarland, et. al.[23], contained any useful information on the structure of fish schools and even this information was rather sketchy.

The work which has been done on the scattering of sound by a single fish is considerably more extensive. Most of the measurements deal with the scattering cross-section of the fish which gives information as to the form of the scattering amplitude $g(\vec{r},a,\omega)$ defined by Eq. 64. However, there is considerable discrepancy and variability among the results reported by the various investigators which warrants a close look at these papers.

First, consider the work of Cushing, et al[2]. In this series of experiments the scattering cross section of fish were measured. The measurements were made at 30 kHz on fish

which were between 15 cm and 100 cm in length. This data allows one to plot a single curve of the scattering cross-section versus the length of the fish, L, for $15 \leq L \leq 100$ cm. However, Cushing first extrapolates this curve to include fish in the range $0 < L \leq 100{,}000$ cm. Then he extrapolates in the frequency domain by plotting curves of the scattering cross-section versus fish length for $0 < L \leq 100{,}000$ cm and for a frequency range from 1 kHz to 10 MHz. All this from one series of measurements of 30 kHz on fish between 15 cm and 100 cm in length! Moreover, there is considerable variability in the scattering cross-section of fish of similar size. For example, the cross-section, σ, from some measurements on perch is given by:

L(cm)	σ(cm^2)
24.0	3.1
22.5	28.7
21.5	71.0
20.5	9.8

To quote Cushing: "No satisfactory explanation could be found to account for these results".

The experimental work of Haslett[24] is much more carefully performed than that of Cushing. Haslett made measurements on guppies and sticklebacks ($0.93 \leq L \leq 6.2$ cm) at frequencies of 360 kHz., 625 kHz and 1.48 MHz. However, like

Cushing, he also extrapolated his measured values.

An examination of the work of Cushing and Haslett shows that the actual measured values were made for fish lengths $L>\lambda_o$, where λ_o is the wavelength in water, and in both papers the results were extrapolated into the region $L<\lambda_o$. However, measurements have been made in the region $L<\lambda_o$ by Coate [25] and Andreeva[16] and they disagree with the extrapolated values of Cushing and Haslett. Thus it must be concluded that the extrapolated curves of Cushing and Haslett do not hold in the region $L<\lambda_o$.

Now two questions must be asked: (1) What causes the difference in the scattering properties of fish in these two regions; (2) Is $L\approx\lambda_o$ a valid criterion for the transition point between these two regions? Unfortunately, the answers to these two questions are not forthcoming and one can only speculate. Weston's opinion is that the return from the fish tissue becomes important when the fish dimensions are comparable to the wavelength, but in fact the height and width dimensions are also significant and the transition really occupies a wide frequency interval. Above this transition, wave interference effects occur between different parts of the fish, different parts of the bladder, and between fish and bladder. Above the transition the tissue and the bladder returns have the same

order of magnitude. Below the transition the bladder echo predominates and below bladder resonance both bladder and tissue are in the Rayleigh scattering region ($\sigma \propto 1/\lambda_o^4$). Weston concludes by stating that "at the low frequencies (i.e., $L/\lambda_o<1$) there is a great need for comprehensive target strength measurements, especially those covering a wide frequency range about the resonance." It would be best to use undamaged live fish, removing uncertainties such as the possibility of bladder shrinkage." This latter remark was made since measurements to date have been performed on dead fish: some freshly killed, some kept in formaldehyde for several weeks before use, some kept frozen for several weeks and thawed before use and others fitted with artificial swim bladders.

Weston's remarks are borne out by the work of Coate in the region $L<\lambda_o$. Coate showed conclusively that for fish of length 20.5 cm and 21.7 cm and for the frequency range of 150 Hz to 1000 Hz, the scattering was due entirely to the air bladder and moreover, for purposes of calculation, the actual elongated bladder could be replaced by an equivalent spherical bladder.

The effect that these results have on the investigation of the long range classification problem posed by fish schools can be summarized as follows: The theory of

multiple scattering is based ultimately on the scattering properties of a single fish. Thus it is imperative to have a reliable model for single fish scattering. For definiteness consider a sonar operating at a frequency of 3000 Hz so that $\lambda_o = 50$ cm. Using the approximate criterion above, it can be said that the scattering from fish of length $L < 50$ cm is due entirely to the bladder. A detailed mathematical model describing scattering by a bladder has been developed[3] and agrees quite well with experimental results. For fish of length $L > 50$ cm the scattering from the fish tissue is as important as that from the bladder. However, no mathematical model has been developed to describe the scattering process in this region. This is unfortunate since these larger fish may cause more of a false alarm problem than the smaller fish. The reason for this is that experimental results indicate that the scattering cross-section increases as the length of the fish increases. However, it is not known if these larger fish school and if they do, just how large the schools actually are.

Thus, for the reasons just explained, we will consider only the case for scattering from bladder fish in the region $L/\lambda_o < 1$. Two geometries will be considered for the boundary surface of the fish school: (1) the fish are contained within a spherical region; (2) the fish are contained within a layer with infinite plane boundaries. In both cases it will be

assumed that the boundary surfaces are smooth. These two geometries are considered first simply because they are the most amenable to solution and understanding the scattering characteristics of obstacles with simple shapes and simple acoustic properties, such as the sphere and the plane, is a necessary stepping stone to an adequate understanding of scattering from obstacles more complicated in shape, composition and structure.

B. SCATTERING FROM A SINGLE FISH

In the region $L/\lambda_o < 1$, scattering is due entirely to the air bladder of the fish and, to a good approximation, the actual elongated bladder can be replaced by an equivalent spherical air bladder for purposes of computation[25]. The scattering characteristics of the bladder are completely specified once the scattering amplitude $g(\vec{r},a,\omega)$, defined by Eq. 64, is known. This function is given by Weston[3] and Andreeva[16] as

$$g(a,\omega) = \frac{a}{\left(\frac{\omega_r^2}{\omega^2} - 1\right) - i\left(\frac{\omega_r}{\omega Q}\right)}, \tag{82}$$

where a is the radius of the equivalent spherical bladder, $\omega/2\pi$ is the frequency of the incident sound wave, $\omega_r/2\pi$ is the resonant frequency of the bladder and Q is a damping factor

which describes the sharpness of resonance. Note that we are assuming that the scattering amplitude $g(a,\omega)$ is independent of the position $\vec{r}$ of the fish. Moreover, we will assume that the scattering amplitude is identical for each fish.

The resonant frequency ω_r of the swim bladder is given by

$$\omega_r = \frac{1}{a}\sqrt{\frac{3\gamma P_h+4\mu_1}{\rho_o}}\ , \tag{83}$$

where ρ_o is the density of water, P_h is the hydrostatic pressure on the bladder and $\gamma = C_p/C_v$, is the ratio of the specific heat at constant pressure, C_p, to the specific heat at constant volume, C_v, of the gas in the bladder. The complex shear modulus, μ, of the bodily tissues of the fish which surround the air bladder is written in terms of its real and imaginary parts as $\mu = \mu_1(1+i\mu_2)$.

The Q-factor, which describes the energy losses incurred in the pulsations of the bladder, can be expressed as a sum of three terms

$$\frac{1}{Q} = \frac{1}{Q_r} + \frac{1}{Q_t} + \frac{1}{Q_f}\ , \tag{84}$$

where Q_r^{-1} describes energy losses due to reradiation, Q_t^{-1}

describes thermal losses and Q_f^{-1} describes viscous losses in the fish tissue.

For radiation damping

$$\frac{1}{Q_r} = \frac{\omega_r a}{c_o} ; \tag{85}$$

For thermal damping

$$\frac{1}{Q_t} = \frac{9P_h(\gamma-1)}{\rho_o \omega_r^2 a^3} \sqrt{\frac{\gamma \nu'}{2\omega_r}} ; \tag{86}$$

For viscous damping

$$\frac{1}{Q_f} = \frac{4\mu_1 \mu_2}{\rho_o \omega_r^2 a^2} . \tag{87}$$

Here c_o is the speed of sound in water and is given by

$$c_o = \sqrt{\frac{B_{ad}}{\rho_o}} , \tag{88}$$

where B_{ad} is the adiabatic bulk modulus of water, and ν' is the thermal diffusivity of the gas in the bladder and is defined by

$$\nu' = \frac{K_t}{\rho_g C_v}, \tag{89}$$

where K_t is the thermal conductivity of the gas and ρ_g is the density of the gas in the bladder.

C. SCATTERING FROM A SPHERICAL SCHOOL OF FISH

Consider a random collection of fish, all with identical scattering properties, contained within a spherical volume of radius A. Let the center of the sphere be located at the origin of the polar coordinate system r, θ, φ (cf. Fig. 2). Since the fish are represented acoustically by their air bladders, the density ρ_f of the medium enclosed by the spherical boundary will be simply the density of the mixture of the water and the air within the bladders. Further, we assume the water to be

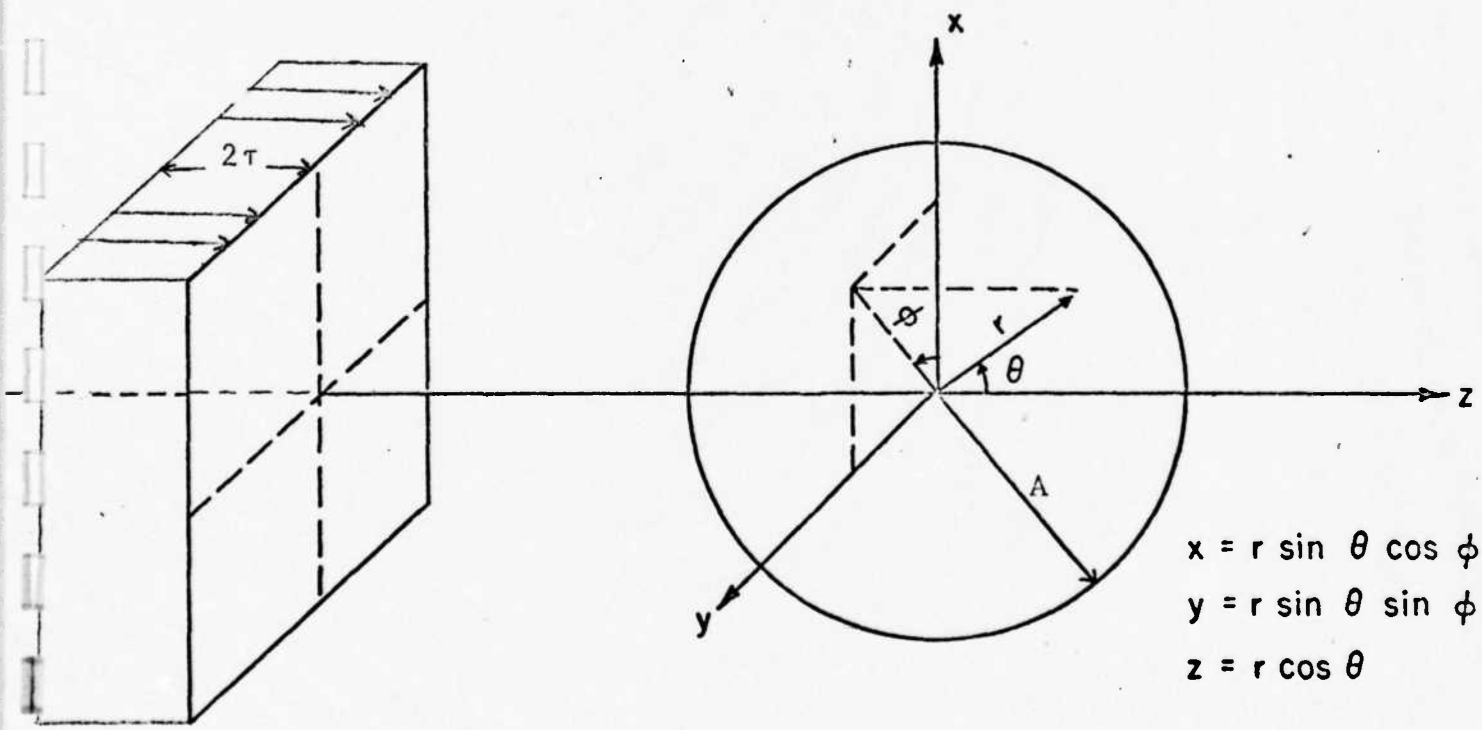

Fig. 2

nonabsorbing and infinite in extent. Let ρ_o denote the constant density of the water and c_o the constant speed of sound in the water as defined by Eq. 88.

A plane acoustic CW-pulse of frequency $\omega_t/2\pi$, pulse length 2τ and acoustic pressure $\Psi^i(\vec{r},t)$, traveling parallel to the polar axis in the positive z-direction, impinges upon the spherical school of fish. (Choosing the incident wave in such a manner eliminates the dependence on φ.) The incident pulse gives rise to an average internal wave $\langle\Psi^f(\vec{r},t)\rangle$ and an average external scattered wave $\langle\Psi^s(\vec{r},t)\rangle$.

The incident CW-pulse can be expressed in the form

$$\Psi^i(\vec{r},t) = \Pi\left[\frac{t - z/c_o}{2\tau}\right] \cos\left(\kappa_t z - \omega_t t\right) , \tag{90}$$

where we have put

$$\kappa_t = \frac{\omega_t}{c_o} \quad . \tag{91}$$

The function $\Pi(\xi)$ is defined by

$$\begin{aligned} \Pi(\xi) &= 1 \text{ , if } |\xi| \leq 1/2 \text{ ,} \\ &= 0 \text{ , if } |\xi| > 1/2. \end{aligned} \tag{92}$$

Now let us compute the Fourier transform of $\Psi^i(\vec{r},t)$. Using Eq. 20, we obtain

$$\psi^i(\vec{r},\omega) = \int_{-\infty}^{\infty} \Psi^i(\vec{r},t)\, e^{i\omega t}\, dt$$

$$\psi^i(\vec{r},\omega) = \int_{-\infty}^{\infty} \Pi\left[\frac{t - z/c_o}{2\tau}\right] \cos\,(\kappa_t z - \omega_t z)\, e^{i\omega t}\, dt$$

$$\psi^i(\vec{r},\omega) = \int_{z/c_o-\tau}^{z/c_o+\tau} \cos\,(\kappa_t z - \omega_t z)\, e^{i\omega t}\, dt$$

$$\psi^i(\vec{r},\omega) = \int_{z/c_o-\tau}^{z/c_o+\tau} \left[\frac{e^{i(\kappa_t z - \omega_t t)} + e^{-i(\kappa_t z - \omega_t t)}}{2}\right] e^{i\omega t}\, dt$$

$$\psi^i(\vec{r},\omega) = \left[\frac{\sin\left[(\omega-\omega_t)\tau\right]}{(\omega-\omega_t)} + \frac{\sin\left[(\omega+\omega_t)\tau\right]}{(\omega+\omega_t)}\right] e^{i\kappa_o z} \tag{93}$$

where we have put

$$\kappa_o = \frac{\omega}{c_o} \tag{94}$$

If we let

$$\phi^i(\omega) = \frac{\sin\left[(\omega-\omega_t)\tau\right]}{(\omega-\omega_t)} + \frac{\sin\left[(\omega+\omega_t)\tau\right]}{(\omega+\omega_t)} \tag{95}$$

we can write Eq. 93 in the form

$$\psi^i(\vec{r},\omega) = \phi^i(\omega)\, e^{i\kappa_o z}. \tag{96}$$

Let $n(\vec{r}',a')\,da'$ be the average number of scatterers per unit volume in the neighborhood of the point $\vec{r}'$ having their scattering parameters lying between a' and $a'+da'$. The function $n(\vec{r}',a')$ is just the number density function introduced in Part I, Section B. Then using Eq. 82 for the scattering amplitude of a single fish $g(a',\omega)$, the scattering amplitude for the school $G(\vec{r}',\omega)$, defined by Eq. 67, becomes

$$G(\vec{r}',\omega) = \int_0^\infty \frac{n(\vec{r}',a')\, a'da'}{\left(\frac{\omega_r^2}{\omega^2} - 1\right) - i\left(\frac{\omega_r}{\omega Q}\right)} . \tag{97}$$

We will assume that there are no fish outside the spherical volume of radius A and that the fish inside the spherical volume are uniformly distributed, i.e., we put

$$\begin{aligned} n(\vec{r}',a') &= n\delta(a'-a), \quad \vec{r}'\epsilon V', \\ &= 0, \quad \vec{r}'\notin V', \end{aligned} \tag{98}$$

where n is a constant representing the number of fish per unit volume and V' is the volume of the sphere of radius A. With this choice for $n(\vec{r}',a')$, Eq. 97 can be integrated to yield

$$G(\vec{r}',\omega) = \frac{na}{\left(\frac{\omega_r^2}{\omega^2} - 1\right) - i\left(\frac{\omega_r}{\omega Q}\right)}, \quad \vec{r}' \in V' \tag{99}$$

$$= 0, \quad \vec{r}' \notin V'.$$

Now the Fourier transform of the average scattered signal $\langle \psi^s(\vec{r},\omega)\rangle$ and the Fourier transform of the average signal within the school $\langle \psi^f(\vec{r},\omega)\rangle$ both satisfy appropriate forms of the Helmholtz equation, Eq. 72.

Since $G(\vec{r}',\omega) = 0$ exterior to the school, we have

$$\nabla^2 \langle \psi^s(\vec{r},\omega)\rangle + \kappa_o^2 \langle \psi^s(\vec{r},\omega)\rangle = 0, \tag{100}$$

where κ_o is given by Eq. 94.

For the wave in the interior of the school we have

$$\nabla^2 \langle \psi^f(\vec{r},\omega)\rangle + k^2(\omega) \langle \psi^f(\vec{r},\omega)\rangle = 0 , \tag{101}$$

where

$$k^2(\omega) = \frac{\omega^2}{c_o^2} + \frac{4\pi na}{\left(\frac{\omega_r^2}{\omega^2} - 1\right) - i\left(\frac{\omega_r}{\omega Q}\right)} . \tag{102}$$

To obtain this expression for $k^2(\omega)$ we have used Eq. 73 and Eq. 99 for $\vec{r}'\epsilon V'$.

Letting

$$k^2 = \gamma + i\,\beta, \tag{103}$$

where γ and β are real, we find from Eq. 102 that

$$\gamma = \frac{\omega^2}{c_o^2} + \frac{4\pi n a \omega^2 (\omega_r^2 - \omega^2)}{(\omega_r^2-\omega^2)^2 + \omega^2\omega_r^2/Q^2} \quad , \tag{104}$$

$$\beta = \frac{4\pi n a \omega^3 \omega_r/Q}{(\omega_r^2 - \omega^2)^2 + \omega^2\omega_r^2/Q^2} \quad . \tag{105}$$

Then

$$\kappa = \left[\frac{1}{2}\left(\gamma + \sqrt{\gamma^2+\beta^2}\right)\right]^{1/2} , \tag{106}$$

and

$$\alpha = \left[\frac{1}{2}\left(-\gamma + \sqrt{\gamma^2+\beta^2}\right)\right]^{1/2} , \tag{107}$$

where κ and α have been defined in Eq. 22. Thus, the wave propagating through the school experiences absorption, which is frequency dependent since α is a function of ω, and dispersion since κ is a function of ω. The phase velocity c_p of the wave is then given by Eq. 24 and in general this will differ from c_o.

We note for future reference that the radial components of the average particle velocities $\langle u^s(\vec{r},\omega)\rangle$, and $\langle u^f(\vec{r},\omega)\rangle$ for the scattered and internal waves respectively, are given in terms of the average pressure fields by the relations[26]:

$$\langle u^s(\vec{r},\omega)\rangle = \frac{1}{i\rho_o\omega} \frac{\partial}{\partial r} \langle \psi^s(\vec{r},\omega)\rangle \ , \tag{108}$$

$$\langle u^f(\vec{r},\omega)\rangle = \frac{1}{i\rho_f\omega} \frac{\partial}{\partial r} \langle \psi^f(\vec{r},\omega)\rangle \ . \tag{109}$$

Likewise for the incident wave we have

$$u^i(\vec{r},\omega) = \frac{1}{i\rho_o\omega} \frac{\partial}{\partial r} \psi^i(\vec{r},\omega) \ . \tag{110}$$

The boundary value problem then consists of finding solutions of the Helmholtz Eqs. 100 and 101 that satisfy the boundary conditions (I) at the surface of the sphere (r=A), the acoustic pressure and (II) the radial components of the particle velocity must be continuous. We now proceed with the solution to this boundary value problem.

The required separated solution [27] in spherical coordinates of Eq. 100 is

$$h_\nu(\kappa_o r)\, P_\nu(\cos\theta) \ , \ \nu = o,1,2,\ldots, \tag{111}$$

where $P_\nu(\cos\theta)$ are the Legendre polynomials [28] and $h_\nu(\kappa_o r)$ is a spherical Hankel function of the first kind defined by

$$h_\nu(\kappa_o r) = j_\nu(\kappa_o r) + i\, n_\nu(\kappa_o r) \ , \tag{112}$$

$j_\nu(\kappa_o r)$ and $n_\nu(\kappa_o r)$ being a spherical Bessel function[29] and a spherical Neumann function [29] respectively.

The required separated solution in spherical coordinates of Eq. 101 is

$$j_\nu(kr)\ P_\nu(\cos\theta) \ , \ \nu=o,1,2,\ldots \ . \tag{113}$$

Due to symmetry these solutions (Eqs. 111 and 113) are independent of φ.

The expansion of the incident wave $\psi^i(r,\theta,\omega)$, given by Eq. 96, in terms of the eigenfunctions of the Helmholtz equation yields[30]

$$\psi^i(r,\theta,\omega) = \Phi^i(\omega)\ e^{i\kappa_o r\cos\theta}$$

$$= \Phi^i(\omega) \sum_{\nu=o}^{\infty} i^\nu(2\nu+1)P_\nu(\cos\theta)j_\nu(\kappa_o r) \ . \tag{114}$$

The scattered wave is expanded in a similar series with coefficients B_ν which are to be determined:

$$\langle \psi^s(r,\theta,\omega)\rangle = \psi^i(\omega) \sum_{\nu=0}^{\infty} i^\nu (2\nu+1) B_\nu \, P_\nu(\cos\theta)\, h_\nu(\kappa_o r) \; . \tag{115}$$

Similarly, for the internal wave, we write

$$\langle \psi^f(r,\theta,\omega)\rangle = \psi^i(\omega) \sum_{\nu=0}^{\infty} i^\nu (2\nu+1) \, A_\nu P_\nu(\cos\theta) j_\nu(kr) \; . \tag{116}$$

The coefficients A_ν and B_ν are found by applying the boundary conditions I and II above. The first boundary condition requires that

$$\psi^i(A,\theta,\omega) + \langle \psi^s(A,\theta,\omega)\rangle = \langle \psi^f(A,\theta,\omega)\rangle \; , \tag{117}$$

and the second boundary condition requires that

$$u^i(A,\theta,\omega) + \langle u^s(A,\theta,\omega)\rangle = \langle u^f(A,\theta,\omega)\rangle \; . \tag{118}$$

Substituting Eqs. 114, 115 and 116 into Eq. 117 yields

$$j_\nu(\kappa_o A) + B_\nu h_\nu(\kappa_o A) = A_\nu j_\nu(kA) . \tag{119}$$

Similarly, substituting Eqs. 114, 115 and 116 into Eq. 118 after having made use of Eqs. 108, 109, and 110

yield

$$j'_\nu(\kappa_o A) + B_\nu h'_\nu(\kappa_o A) = Z(\omega) \; A_\nu j'_\nu(kA) \; , \tag{120}$$

where the prime denotes differentiation with respect to the argument and $Z(\omega)$ is the ratio of the acoustic impedance of water Z_o to that of the fish school $Z_f(\omega)$, i.e.,

$$Z(\omega) = \frac{Z_o}{Z_f(\omega)} \; , \tag{121}$$

where

$$Z_o = \rho_o c_o \; , \tag{122}$$

$$Z_f(\omega) = \rho_f c_f \; , \tag{123}$$

$$c_f = \omega/k \; . \tag{124}$$

Note that since k is complex, the speed of sound in the fish school c_f is complex and consequently the ratio of impedances $Z(\omega)$ is complex.

Solving Eqs. 119 and 120 simultaneously, we obtain

$$B_\nu = - \left[\frac{j_\nu(kA) j'_\nu(\kappa_o A) - Z(\omega) \; j_\nu(\kappa_o A) j'_\nu(kA)}{j_\nu(kA) h'_\nu(\kappa_o A) - Z(\omega) j'_\nu(kA) h_\nu(\kappa_o A)} \right] . \tag{125}$$

Next let us eliminate the derivatives of the Bessel functions in Eq. 125 by use of the well known relation

$$f'_\nu(\xi) = \frac{\nu}{\xi} f_\nu(\xi) - f_{\nu+1}(\xi) , \tag{126}$$

where $f_\nu(\xi)$ can be $j_\nu(\xi)$ or $n_\nu(\xi)$.

Using this result, we can write the expression for B_ν in the form

$$B_\nu = \frac{N_\nu}{M_\nu} , \tag{127}$$

where

$$N_\nu = \left[\nu(\kappa_o A)^{-1}(1-h) j_\nu(\kappa_o A) - j_{\nu+1}(\kappa_o A)\right] j_\nu(kA)$$

$$+ Z(\omega)\, j_\nu(\kappa_o A)\, j_{\nu+1}(kA) , \tag{128}$$

$$M_\nu = \left[\nu(\kappa_o A)^{-1}(1-h)\, h_\nu(\kappa_o A) - h_{\nu+1}(\kappa_o A)\right] j_\nu(kA)$$

$$+ Z(\omega)\, h_\nu(\kappa_o A)\, j_{\nu+1}(kA) , \tag{129}$$

and we have put

$$h = \rho_o/\rho_f . \tag{130}$$

Since we want to consider the scattered wave only

at large distances from the fish school, we can use the following asymptotic expansion for the Hankel function

$$h_\nu(\kappa_o r) \longrightarrow \frac{i^{-\nu} e^{i\kappa_o r}}{i\kappa_o r}, \quad (\kappa_o r \longrightarrow \infty) . \tag{131}$$

Thus the expression (Eq. 115) for the scattered wave becomes in the asymptotic region

$$\left\langle \psi^s(r,\theta,\omega) \right\rangle = \Phi^i(\omega) S(\theta,\omega) \frac{e^{i\kappa_o r}}{r} , \tag{132}$$

where

$$S(\theta,\omega) = (i\kappa_o)^{-1} \sum_{\nu=o}^{\infty} (2\nu+1) B_\nu(\omega) P_\nu(\cos\theta). \tag{133}$$

Consequently, the expression for the scattered signal becomes

$$\left\langle \Psi^s(r,\theta,t) \right\rangle = \frac{1}{\pi} \operatorname{Re} \int_o^\infty \Phi^i(\omega)\, S(\theta,\omega) \frac{e^{-i\omega(t-r/c_o)}}{r} d\omega . \tag{134}$$

Now if we write

$$S(\theta,\omega) = S_1 + i\, S_2 , \tag{135}$$

where S_1 and S_2 are real, Eq. 134 becomes

$$\left\langle \Psi^s(r,\theta,t)\right\rangle = \frac{1}{\pi r}\int_0^\infty \Phi^i(\omega)\left[S_1(\theta,\omega)\cos\omega t^* + S_2(\theta,\omega)\sin\omega t^*\right]d\omega, \tag{136}$$

where t^* is the retarded time

$$t^* = t - r/c_o\ . \tag{137}$$

Finally, let us calculate the scattered signal in the Born approximation. The Fourier transform of the signal is given by Eq. 76

$$\left\langle \psi^s(r,\theta,\omega)\right\rangle = \int G(\vec{r}\,',\omega)\,\psi^i(\vec{r}\,',\omega)\,\frac{e^{i\kappa_o|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\,d\vec{r}\,'\ . \tag{138}$$

Let (r,θ,φ) be the spherical coordinates of $\vec{r}$ and (r',θ',φ') those of $\vec{r}\,'$. Further, let Θ be the angle between $\vec{r}$ and $\vec{r}\,'$ (c_f. Fig. 3).

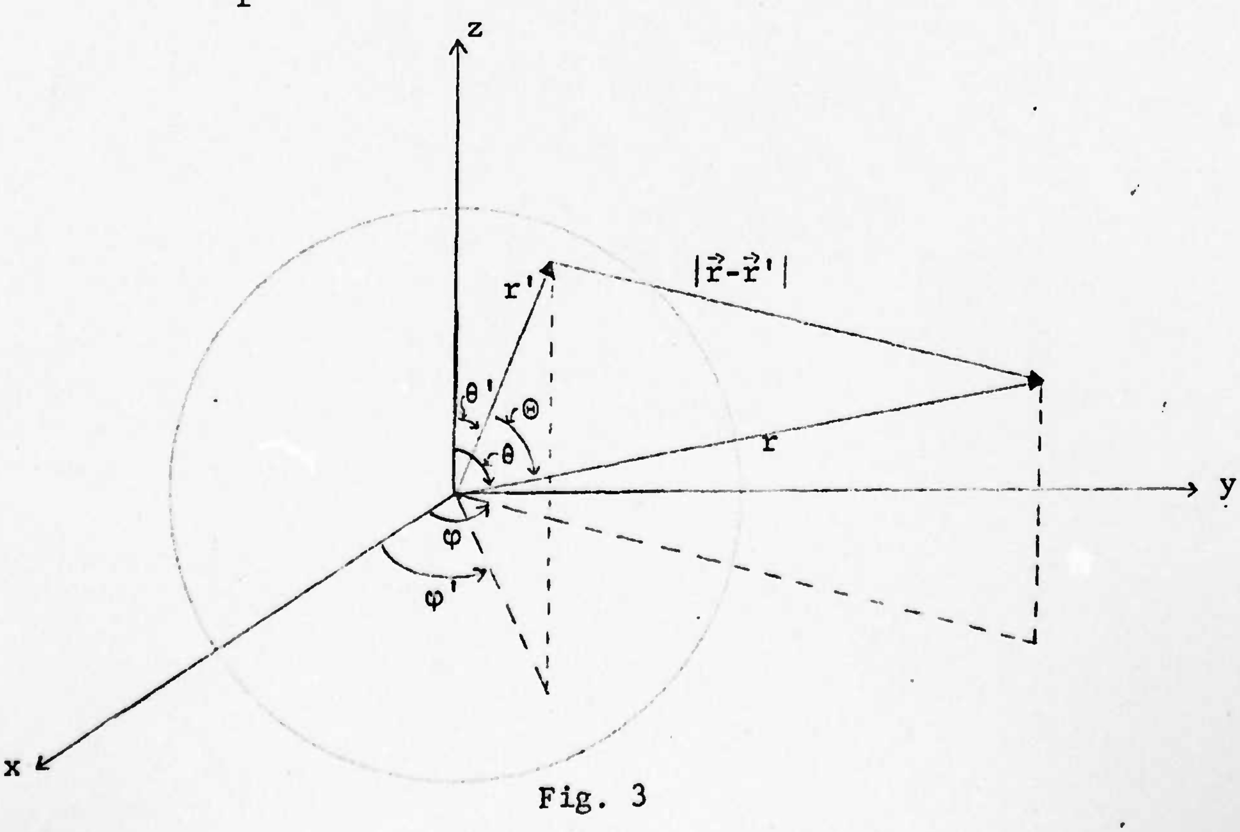

Fig. 3

Using Eq. 99 for $G(\vec{r}',\omega)$ and Eq. 96 for $\psi^i(\vec{r}',\omega)$, the expression for the scattered wave given by Eq. 138 becomes

$$\langle\psi^s(r,\theta,\omega)\rangle = G(\omega)\Phi^i(\omega)\int_0^A dr'\int_0^\pi d\theta'\int_0^{2\pi} d\varphi'\, r'^2 \sin\theta'$$

$$\times\, e^{iK_o r'\cos\theta'}\,\frac{e^{iK_o|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \tag{139}$$

Now

$$|\vec{r}-\vec{r}'| = \left[r^2 + r'^2 - 2rr'\cos\Theta\right]^{1/2}, \tag{140}$$

where

$$\cos\Theta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi') . \tag{141}$$

Since we want to consider the scattered wave only in the far field, we have $r' \ll r$. Consequently, we can expand Eq. 140 retaining only first order terms:

$$|\vec{r}-\vec{r}'| = r\left[1 + \frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\Theta\right]^{1/2}$$

$$\approx r\left[1 - \frac{r'}{r}\cos\Theta\right]$$

$$\approx r - r'\cos\Theta \tag{142}$$

While it is necessary to retain the first order term $r'\cos\Theta$ as far as the factor $|\vec{r}-\vec{r}'|$ which appears in the phase is concerned, it is not necessary for the factor $|\vec{r}-\vec{r}'|$ which appears in the amplitude. Consequently for the

amplitude we can write $|\vec{r}-\vec{r}'| \approx r$.

Moreover, we wish only to consider the back-scattered wave in the direction $\theta = \pi$, so that $\cos \Theta = - \cos \theta'$ from Eq. 141. Collecting all of these results allows us to write Eq. 139 as

$$\langle \psi^s(r,\pi,\omega)\rangle = G(\omega)\Phi^i(\omega)\left[\frac{e^{i\kappa_o r}}{r}\right]\int_0^A dr' \int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi' r'^2 \sin\theta' \, e^{2i\kappa_o r'\cos\theta'}$$

$$\langle \psi^s(r,\pi,\omega)\rangle = 2\pi\, G(\omega)\Phi^i(\omega)\, \frac{e^{i\kappa_o r}}{r} \int_0^A dr' \int_0^{\pi} d\theta' r'^2 \sin\theta' \, e^{2i\kappa_o r'\cos\theta'}$$

$$\langle \psi^s(r,\pi,\omega)\rangle = 2\pi G(\omega)\, \Phi^i(\omega)\, \frac{e^{i\kappa_o r}}{\kappa_o r} \int_0^A \sin(2\kappa_o r')\, r'dr'$$

$$\langle \psi^s(r,\pi,\omega)\rangle = \frac{\pi G(\omega)\Phi^i(\omega)}{2\kappa_o^3}\left[\sin(\kappa_o D)-\kappa_o D\cos(\kappa_o D)\right]\frac{e^{i\kappa_o r}}{r} \qquad (143)$$

where D = 2A.

The scattered pulse in the Born approximation then becomes

$$\langle \Psi^s(r,\pi,t)\rangle = \frac{1}{\pi}\,\mathrm{Re}\int_0^{\infty} \langle \psi^s(r,\pi,\omega)\rangle\, e^{-i\omega t}\, d\omega$$

$$\langle \Psi^s(r,\pi,t^*)\rangle = \frac{1}{2r}\,\mathrm{Re}\int_0^{\infty} \frac{G(\omega)\Phi^i(\omega)}{\kappa_o^3}\left[\sin(\kappa_o D)-\kappa_o D\cos(\kappa_o D)\right] e^{-i\omega t^*} d\omega \qquad (144)$$

where, as before, t* is the retarded time defined by Eq. 137.

If we let

$$G(\omega) = G_1 + i\, G_2 \;, \tag{145}$$

where from Eq. 99, we find that

$$G_1 = \frac{na\omega^2\,(\omega_r^2-\omega^2)}{(\omega_r^2 - \omega^2)^2 + \omega_r^2\omega^2/Q^2} \;, \tag{146}$$

$$G_2 = \frac{na\omega^3\,\omega_r/Q}{(\omega_r^2-\omega^2)^2 + \omega_r^2\omega^2/Q^2} \;, \tag{147}$$

then we can write Eq. 144 as

$$\left\langle \Psi^s(r,\Pi,t^*) \right\rangle = \frac{1}{2r} \int_0^\infty \frac{\Phi^i(\omega)}{\kappa_o^3} \left[\sin(\kappa_o D) - \kappa_o D \cos(\kappa_o D)\right]$$

$$\times \left[G_1 \cos \omega t^* + G_2 \sin \omega t^*\right] d\omega \;. \tag{148}$$

This is the average of the scattered signal in the Born approximation. Note that the scattered signal is directly proportional to the average number of scatterers per unit volume n as can be seen from Eqs 146 and 147.

D. SCATTERING FROM A LAYER OF FISH WITH PLANE BOUNDARIES

Consider a random collection of fish, all with identical scattering properties and uniformly distributed within a volume bounded by two parallel, infinite planes. Let us choose a coordinate system so that the planes are parallel to the xy-plane and let one plane intersect the z-axis at $z = h$ and the other intersect at $z = h + \Delta h$ $(\Delta h > 0)$. The geometry is illustrated in Fig. 4.

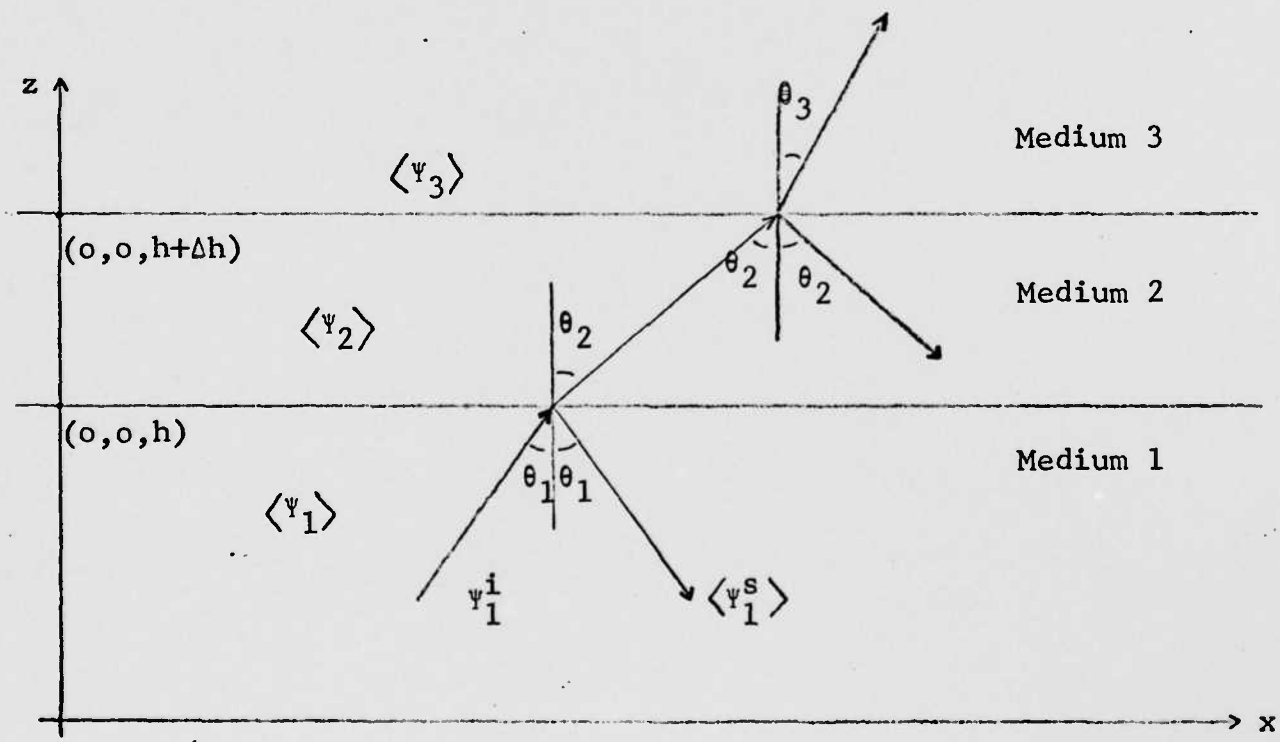

Fig. 4

In reference to Fig. 4, media 1 and 3 are water while medium 2 is a mixture of water and fish. Since the same assumptions will be made for the layer of fish as were made for the spherical school, full use will be made of the results in Section C without restating or rederiving them here.

Let ρ_i be the density and c_i the sound velocity in the i^{th} medium. Then, from what was said above, $\rho_1 = \rho_3 = \rho_o$, $\rho_2 = \rho_f$, $c_1 = c_3 = c_o$ and $c_2 = c_f$.

Let $\Psi_1^i(\vec{r},t)$ be a plane pulse, traveling in medium 1, which is incident on the layer. Since the boundary of the layer is a plane, it is more convenient to use the Fourier plane wave expansion for $\Psi_1^i(\vec{r},t)$ than the general expansion given by Eq. 19. To obtain the plane wave expansion from the general expansion (Eq. 19), one simply lets the Fourier transform $\psi_1^i(\vec{r},\omega)$ be a plane wave:

$$\psi_1^i(\vec{r},\omega) = \Phi^i(\omega)\, e^{i\vec{k}_1 \cdot \vec{r}} , \tag{149}$$

where $\Phi^i(\omega)$ is the amplitude of the wave and depends only on the frequency ω. For the case of a CW-pulse, $\Phi^i(\omega)$ is given by Eq. 95.

The vector $\vec{k}_1$ is called the wave vector. Its direction is that of the normal to the wavefront and its magnitude, denoted k_1, is given by

$$k_1 = \kappa_o = \frac{\omega}{c_o} = \frac{2\pi}{\lambda_o} \tag{150}$$

Let $\hat{n}$ be the unit normal vector to the wavefront and let n_x, n_y, n_z be the components of $\hat{n}$ along the x, y and z-axes, respectively. Since $\hat{n}$ is a unit vector, the components must satisfy the relation

$$n_x^2 + n_y^2 + n_z^2 = 1 \quad . \tag{151}$$

Now we can write $\vec{k}_1$ in the form

$$\vec{k}_1 = k_1 \hat{n} \ , \tag{152}$$

and the x, y and z components of $\vec{k}_1$, denoted k_{1x}, k_{1y}, k_{1z}, in the form

$$k_{1x} = k_1 n_x, \quad k_{1y} = k_1 n_y, \quad k_{1z} = k_1 n_z \ , \tag{152}$$

so that

$$k_{1x}^2 + k_{1y}^2 + k_{1z}^2 = k_1^2 \, (n_x^2 + n_y^2 + n_z^2) = k_1^2 \ . \tag{153}$$

Now consider the situation illustrated in Fig. 4 where the unit normal $\hat{n}$ to the incident wavefront lies in the xz-plane and makes an angle θ_1, with the positive z-axis. In this case $n_x = \sin\theta_1$, $n_y = 0$, $n_z = \cos\theta_1$, and Eq. 149 becomes

$$\begin{aligned} \psi_1^i (\vec{r},\omega) &= \phi^i(\omega)\, e^{i(k_{1x}x + k_{1y}y + k_{1z}\, z)} \\ &= \phi^i(\omega)\, e^{ik_1(n_x x + n_z z)} \\ &= \phi^i(\omega)\, e^{ik_1\,(x \sin\theta_1 + z \cos\theta_1)} \end{aligned} \tag{154}$$

Thus the Fourier integral representation (Eq. 19) for $\Psi_1^i(\vec{r},t)$ becomes

$$\Psi_1^i(x,z,t) = \frac{1}{\pi} \mathrm{Re} \int_0^\infty \Phi^i(\omega)\, e^{ik_1(x \sin\theta_1 + z\cos\theta_1) - i\omega t} d\omega \,. \tag{155}$$

A similar Fourier plane wave expansion can be made for the average of the signal scattered back into medium 1:

$$\langle \Psi_1^s(x,z,t) \rangle = \frac{1}{\pi} \mathrm{Re} \int_0^\infty R(\omega) e^{ik_1(x \sin\theta_1 - z\cos\theta_1) - i\omega t} d\omega \,, \tag{156}$$

where $R(\omega)$ is the reflection coefficient to be determined by boundary conditions.

Let $\langle \Psi_1(x,z,t) \rangle$ denote the total sound field in medium 1 so that

$$\langle \Psi_1(x,z,t) \rangle = \Psi_1^i(x,z,t) + \langle \Psi_1^s(x,z,t) \rangle \,. \tag{157}$$

We can represent the average of the total sound field in medium 2, $\langle \Psi_2(x,z,t) \rangle$, and in medium 3, $\langle \Psi_3(x,z,t) \rangle$, in a similar manner:

$$\langle \Psi_2(x,z,t) \rangle = \frac{1}{\pi} \mathrm{Re} \int_0^\infty A_2(\omega)\, e^{ik_2(x \sin\theta_2 + z\cos\theta_2) - i\omega t} d\omega$$

$$+ \frac{1}{\pi} \mathrm{Re} \int_0^\infty B_2(\omega)\, e^{ik_2(x \sin\theta_2 - z\cos\theta_2) - i\omega t} d\omega \,, \tag{158}$$

$$\left\langle \Psi_3(x, z, t) \right\rangle = \frac{1}{\pi} \operatorname{Re} \int_0^\infty T(\omega)\, e^{ik_3(x \sin \theta_3 + z \cos \theta_3) - i\omega t} d\omega \,. \tag{159}$$

Here $k_2 = k$, $k_3 = \kappa_o$ and θ_2 is the angle of refraction in medium 2 while θ_3 is the angle of refraction in medium 3 (Cf. Fig. 4). The amplitudes $A_2(\omega)$, $B_2(\omega)$ and $T(\omega)$ of the respective waves are to be determined by the boundary conditions. The first integral in Eq. 152 represents the total sound field which is propagating in the direction with $k_{2z} > 0$ and the second integral represents the total sound field which is propagating in the direction with $k_{2z} < 0$.

Since the normal to the boundaries is parallel to the z-axis, the normal components of the particle velocities are given by

$$\left\langle u_m(x,z,\omega) \right\rangle = \frac{1}{i \rho_m \omega} \frac{\partial}{\partial z} \left\langle \psi_m(x,z,\omega) \right\rangle \tag{160}$$

for m = 1,2,3.

The boundary conditions are then that the pressure and the normal components of the particle velocities must be continuous across both planes comprising the boundary.

Continuity of pressure at z = h:

$$\left\langle \psi_1 (x,h,\omega) \right\rangle = \left\langle \psi_2(x,h,\omega) \right\rangle \,. \tag{161}$$

Continuity of pressure at $z = h + \Delta h$:

$$\left\langle \psi_2(x,h + \Delta h, \omega) \right\rangle = \left\langle \psi_3(x,h + \Delta h,\omega) \right\rangle \,. \tag{162}$$

Continuity of the normal component of particle velocity at $z = h$:

$$\langle u_1(x,h,\omega)\rangle = \langle u_2(x,h,\omega)\rangle \ . \tag{163}$$

Continuity of the normal component of particle velocity at $z = h + \Delta h$:

$$\langle u_2(x,h+\Delta h,\omega)\rangle = \langle u_3(x,h+\Delta h,\omega)\rangle \ . \tag{164}$$

Using Eqs. 155-158, the first boundary condition (Eq. 161) becomes

$$\phi^i(\omega)\, e^{ik_1(x \sin\theta_1 + h\cos\theta_1)} + R(\omega)\, e^{ik_1(x\sin\theta_1 - h\cos\theta_1)}$$

$$= A_2(\omega)\, e^{ik_2(x\sin\theta_2 + h\cos\theta_2)} + B_2(\omega)\, e^{ik_2(x\sin\theta_2 - h\cos\theta_2)} \tag{165}$$

Rearranging the terms in this equation, allows us to write it as

$$e^{ix(k_1\sin\theta_1 - k_2\sin\theta_2)} = \left[A_2 e^{ik_2 h\cos\theta_2} + B_2 e^{ik_2 h\cos\theta_2}\right]$$

$$\times \left[\phi^i e^{ik_1 h\cos\theta_1} + Re^{-ik_1 h\cos\theta_1}\right]^{-1} \ . \tag{166}$$

Since the right side of this last equation is independent of x, the left side must also be independent of x. Consequently, we must have

$$k_1 \sin \theta_1 = k_2 \sin \theta_2 \tag{167}$$

This is just Snell's law of refraction for media 1 and 2.

Using Eqs. 158 and 159, the second boundary condition (Eq. 162) becomes

$$A_2(\omega)\, e^{ik_2(x \sin \theta_2 + h \cos \theta_2 + \Delta h \cos \theta_2)}$$

$$+ B_2(\omega)\, e^{ik_2(x \sin \theta_2 - h \cos \theta_2 - \Delta h \cos \theta_2)}$$

$$= T(\omega)\, e^{ik_3(x \sin \theta_3 + h \cos \theta_3 + \Delta h \cos \theta_3)} \quad . \tag{168}$$

Rearranging this last equation, we get

$$e^{ix(k_2 \sin \theta_2 - k_3 \sin \theta_3)} = \left[A_2 e^{ik_2(h \cos \theta_2 + \Delta h \cos \theta_2)} + B_2 e^{-ik_2(h \cos \theta_2 + \Delta h \cos \theta_2)}\right]^{-1} A_3 e^{ik_3(h \cos \theta_3 + \Delta h \cos \theta_3)} . \tag{169}$$

Again, the right side of this last equation is independent of x, so we must have

$$k_2 \sin \theta_2 = k_3 \sin \theta_3 \; . \tag{170}$$

This is Snell's law of refraction for media 2 and 3.

Combining Eqs. 167 and 170, yield

$$k_1 \sin \theta_1 = k_2 \sin \theta_2 = k_3 \sin \theta_3 \tag{171}$$

Using Eqs. 155-158 and 160, the third boundary condition (Eq. 163) becomes

$$\frac{1}{i\omega\rho_1}\left[ik_1\cos(\theta_1)\phi^i\, e^{ik_1(x\sin\theta_1 + h\cos\theta_1)} - ik_1\cos(\theta_1)R\, e^{ik_1(x\sin\theta_1 - h\cos\theta_1)}\right]$$

$$= \frac{1}{i\omega\rho_2}\left[ik_2\cos(\theta_2)A_2\, e^{ik_2(x\sin\theta_2 + h\cos\theta_2)} - ik_2\cos(\theta_2)B_2\, e^{ik_2(x\sin\theta_2 - h\cos\theta_2)}\right]. \tag{172}$$

Finally, using Eqs. 158-160, the fourth boundary condition (eq. 164) becomes

$$\frac{1}{i\omega\rho_2}\left[ik_2\cos(\theta_2)A_2\, e^{ik_2(x\sin\theta_2 + h\cos\theta_2 + \Delta h\cos\theta_2)} - i\,k_2\cos(\theta_2)B_2\, e^{ik_2(x\sin\theta_2 - h\cos\theta_2 - \Delta h\cos\theta_2)}\right]$$

$$= \frac{1}{i\omega\rho_3}\left[ik_3\cos(\theta_3)T\, e^{ik_3(x\sin\theta_3 + h\cos\theta_3 + \Delta h\cos\theta_3)}\right]. \tag{173}$$

If we use Eq. 171 and the fact that $k_i = \omega/c_i$, Eqs. 165, 168, 172 and 173 can be simplified to

$$\phi^i e^{ik_1 h \cos\theta_1} + R e^{-ik_1 h \cos\theta_1} = A_2 e^{ik_2 h \cos\theta_2} + B_2 e^{-ik_2 h \cos\theta_2} , \tag{174}$$

$$A_2 e^{ik_2(h+\Delta h)\cos\theta_2} + B_2 e^{-ik_2(h+\Delta h)\cos\theta_2}$$

$$= T e^{ik_3(h\cos\theta_3 + \Delta h\cos\theta_3)} , \tag{175}$$

$$\frac{\phi^i}{Z_1} e^{ik_1 h\cos\theta_1} - \frac{R}{Z_1} e^{-ik_1 h\cos\theta_1} = \frac{A_2}{Z_2} e^{ik_2 h\cos\theta_2}$$

$$- \frac{B_2}{Z_2} e^{-ik_2 h\cos\theta_2} , \tag{176}$$

$$\frac{A_2}{Z_2} e^{ik_2(h+\Delta h)\cos\theta_2} - \frac{B_2}{Z_2} e^{-ik_2(h+\Delta h)\cos\theta_2}$$

$$= \frac{T}{Z_3} e^{ik_3(h+\Delta h)\cos\theta_3} , \tag{177}$$

where we have defined the impedance Z_i as

$$Z_i = \frac{\rho_i c_i}{\cos\theta_i} . \tag{178}$$

Eqs. 175-178 represent a system of four equations for the four unknown amplitude functions R, A_2, B_2 and T. When they are solved simultaneously for R they yield

$$R(\omega) = \Phi^i(\omega)\left[\frac{V_{21} + V_{32}\, e^{2ik_2\Delta h \cos\theta_2}}{1 + V_{21}V_{32}\, e^{2ik_2\Delta h \cos\theta_2}}\right] e^{2ik_1 h \cos\theta_1}\ , \qquad (179)$$

where we have put

$$V_{ij} = \frac{Z_i - Z_j}{Z_i + Z_i}\ . \qquad (180)$$

Since we are only interested in the reflected wave, the other amplitudes will not be determined.

The result, Eq. 179, can be put into a more convenient form for computation. If we let

$$D_{ij} = \frac{2\, Z_j}{Z_i + Z_j}\ , \qquad (181)$$

and expand the denominator of Eq. 179 in a power series, we obtain

$$\frac{R(\omega)\, e^{-2ik_1 h \cos\theta_1}}{\Phi^i(\omega)} = V_{21} + D_{12}\, V_{32}\, D_{21}\, e^{2ik_2 \Delta h \cos\theta_2}$$

$$+ D_{12}\, V_{32}\, V_{12}\, V_{32}\, D_{21}\, e^{4ik_2 \Delta h \cos\theta_2} + \ldots, \qquad (182)$$

or written more compactly,

$$\frac{R(\omega)\, e^{-2ik_1 h \cos\theta_1}}{\Phi^i(\omega)} = V_{21} + D_{12}\, V_{32}\, D_{21}\, e^{2ik_2 \Delta h \cos\theta_2}$$

$$\times \sum_{n=o}^{\infty} \left[V_{12} V_{32}\, e^{2ik_2 \Delta h \cos\theta_2} \right]^n \qquad (183)$$

This last form (Eq. 183) for the reflection coefficient has an interesting physical interpretation.[31] It is the resolution of the total reflected wave into individual waves which represent the various multiple orders of reflection at the boundaries of the layer. The coefficient V_{ij} can be interpreted as the reflection coefficient of the boundary between the i^{th} and j^{th} media for a wave that is traveling from the j^{th} medium to the i^{th} medium. Likewise, the coefficient D_{ij} can be interpreted as the transmission coefficient of the boundary between the i^{th} and j^{th} media for a wave that is traveling from the j^{th} medium to the i^{th} medium.

Using this interpretation of V_{ij} and D_{ij}, let us consider the individual terms in Eq. 182. The first term V_{21} is the complex amplitude of a wave that has been reflected once from the boundary between media 1 and 2. The second term $D_{21}\ V_{32}\ D_{21}\ e^{2ik_2 \Delta h \cos \theta_2}$ is the complex amplitude of the wave which has been transmitted from medium 1 to medium 2 (D_{21}), passed through medium 2, was reflected at the boundary between media 2 and 3 (V_{32}), passed back through the layer and back into medium 1 again (D_{12}). The additional phase shift which occurs in the exponential is simply due to the slant distance across the layer which is just ($\Delta h \cos \theta_2$). The succeeding terms may be interpreted similarly.

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